22nd CZECH-JAPAN SEMINAR ON DATA ANALYSIS AND DECISION MAKING

Nový Světlov, September 25-28, 2019

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Foreword

The history of the series of the Czech-Japan seminars started in 1999. Thus, it is now more than 20 years ago when the first Czech-Japan Seminar on Data Analysis and Decision Making under Uncertainty was held in JAIST, Hokuriku. Since that time, these seminars were held in eleven splendid places in Japan, offering the Czech participants possibility to discover different parts of the Japanese islands. In reciprocity, it was the goal of the Czech partners organizing the past ten seminars to show the beauty of Czechia to Japanese colleagues, who, during the long Japan–Czech cooperation, became our close friends. This is also why the seminar has never visited one place two times.

The 22nd seminar takes place in a castle Nový Světlov located in the far east corner of Czechia - close to the border with Slovakia. Originally, the castle was founded in the fifteenth century to protect the borders of the Czech Kingdom. Though the castle has more than five hundred years history, it was several times reconstructed so that nowadays it offers comfort to the seminar participants. They will also have a possibility to visit a close spa town Luhačovice with a spring of heavily mineralized water Vincentka famous for a positive effect on diseases of vocal cords and breathing pathways.

The Proceedings you are holding in your hands contain contributions to be presented in Nový Světlov. When reading its content, keep in mind the working character of the Czech-Japanese seminars. The participants are primarily invited to present new open problems or to introduce ongoing projects with unfinished results. These presentations inspire informal discussions which were always an integral part of the Czech Japanese seminars, and which we also expect in this year 2019.

On behalf of the Organizing and Programme Committees

Masahiro Inuiguchi, Radim Jiroušek and *Václav Kratochvíl*

List of Content

On stability and interpretability of mixture models of wage distributions

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Abstract

Statistical analysis of the development of the wage and income distribution is a crucial precondition for economic modeling of the labor market processes. A good model that is able to make good predictions of the future wage distributions is necessary for various socio-economic considerations. When considering a mixture model, the key question is what should be the components of the model. A mixture model can be created using two different approaches. In the first approach, the model is created manually, based on user-specified components, their weights and parameters of the components. In the second approach, the model is created completely from data. The partitioning will be done automatically, we need not even to specify the number of components.We will compare both approaches addressing the issues of stability and interpretability of the created models. Data concerning the wages of Czech employees collected for more than twenty years will be used in our study. We will create models for each particular year and evaluate the changes of weights of the components (the stability issue) and the meaning of the components (the interpretability issue).

1 Introduction

Statistical analysis of the development of the wage and income distribution is a crucial precondition for economic modeling of the labor market processes. A good

model that is able to make good predictions of the future wage distributions is necessary for various socio-economic considerations. Our previous work shows that various probabilistic distributions can be used to model the empirical wage distribution. Among single probabilistic distributions, the three-parameter Log-logistic distribution gives the best results, followed by the three-parameter Log-normal distribution. But as the empirical wage distribution becomes less smooth over time, mixture models, in particular normal mixture, give good results now and are more promising for the future.

When considering a mixture model, the key question is what should be the components of the model. A mixture model can be created using two different approaches. In the first approach, the model is created manually, based on user-specified components, their weights and parameters of the components. We can e.g. split the population into group of men, group of women and use their proportion in the population, and the mean values and standard deviations of wages in both groups as the corresponding parameters of the model. In the second approach, the model is created completely from data. The partitioning will be done automatically, we need not even to specify the number of components.

We will compare both approaches addressing the issues of stability and interpretability of the created models. Data concerning the wages of Czech employees collected for more than twenty years will be used in our study. We will create models for each particular year and evaluate the changes of weights of the components (the stability issue) and the meaning of the components (the interpretability issue). Our expectation is, that the models created from data will be more accurate but less stable and less interpretable.

2 Wage Data

We work with time series of wages in Czech Republic (CR) covering the years 1995 - 2018. Our data are in the form of an interval frequency distribution table; the wage values are divided into intervals with widths of 500 CZK. The data is always taken from the second calendar quarter in each year because of the long-term stability of the working hours' volume in that quarter. The amount of data gradually increases from the sample size of about 300 000 in 1995 to more than two million in 2018. The source of our data is the consultant company Trexima (http://www.trexima.cz); this company collects the wage data for the Czech Ministry of Labor and Social Affairs. Table in Fig. 1, shows basic characteristics of the wage distributions for each of this year: number of observations, average value, standard deviation, the first decile (10% quantile, D1), lower quartile (25% quantile, Q1), median (50% quantile), third quartile (75% quantile, Q3), and ninth decile (90% quantile, D9). Fig. 2 visualizes the distribution of wages from these data. The curves shown in the graph are produced by connecting points of frequency for 500 CZK intervals, there is no method of empirical distribution smoothing applied. The figure clearly shows that the empirical wage distributions:

- are bounded by minimum wages
- are skewed, and
- change over time as the average value increases, the variability increases and the distributions become less smooth.

We also observed, that the proportion of high wages (the wages above 100,000 CZK, i.e. approx. 4,000 EUR) is growing.

Year	No.obs.	Average	StanDev	D1	Q1	Median	Q3	D9
1995	321,277	8,311	4,133	4,879	5,963	7,500	9,691	12,314
1996	405,138	9,962	5,393	5,645	7,047	8,956	11,505	14,748
1997	622,505	11,322	6,490	6,178	7,910	10,171	13,083	16,774
1998	953,691	12,026	8,261	6,287	8,114	10,563	13,801	17,911
1999	1,024,898	12,982	8,262	6,894	8,859	11,506	14,911	19,499
2000	1,053,536	13,541	9,651	6,981	9,077	11,860	15,570	20,435
2001	1,075,875	14,743	10,372	7,693	9,870	12,901	16,794	22,234
2002	1,107,991	15,964	12,994	8,181	10,564	13,857	18,058	24,003
2003	1,230,282	17,748	13,504	9,143	11,829	15,519	20,070	26,271
2004	1,680,800	17,759	13,062	9,185	12,073	15,789	20,168	26,143
2005	1,818,369	18,640	13,796	9,371	12,403	16,432	21,376	27,754
2006	1,976,571	19,526	17,696	9,710	12,882	17,143	22,192	28,828
2007	2,059,416	20,953	18,055	10,381	13,659	18,185	23,602	31,257
2008	2,079,765	22,338	20,714	11,060	14,583	19,267	25,094	33,306
2009	1,933,772	23,418	19,014	11,681	15,339	20,138	26,241	35,093
2010	1,956,702	24,077	19,316	12,084	15,778	20,753	27,009	36,143
2011	1,973,468	24,484	24,802	12,199	15,996	21,020	27,225	36,677
2012	1,999,934	24,829	20,109	12,255	16,281	21,319	27,583	37,328
2013	2,015,903	25,448	20,564	12,416	16,595	21,779	28,322	38,598
2014	2,056,133	25,728	19,612	12,570	16,821	22,074	28,794	39,182
2015	2,098,854	26,369	19,903	12,978	17,290	22,658	29,566	40,162
2016	2,119,396	27,668	20,478	13,944	18,391	23,757	30,963	42,026
2017	2,185,573	29,166	20,749	14,982	19,547	25,135	32,610	44,334
2018	2,237,108	31,992	21,576	16,926	21,714	27,738	35,878	48,137

Figure 1: Wage data characteristics.

3 Mixture Models of Wage Distributions

Various probabilistic distributions can be used to model the empirical wage distribution. Our previous work shows, that three parameter log-logistic distribution gives the best results (out of about 50 different probabilistic distributions available in the EasyFit system) when modeling the empirical wage distribution in CR for the years 2000-2017 [5, 7]. But as the empirical wage distribution becomes less

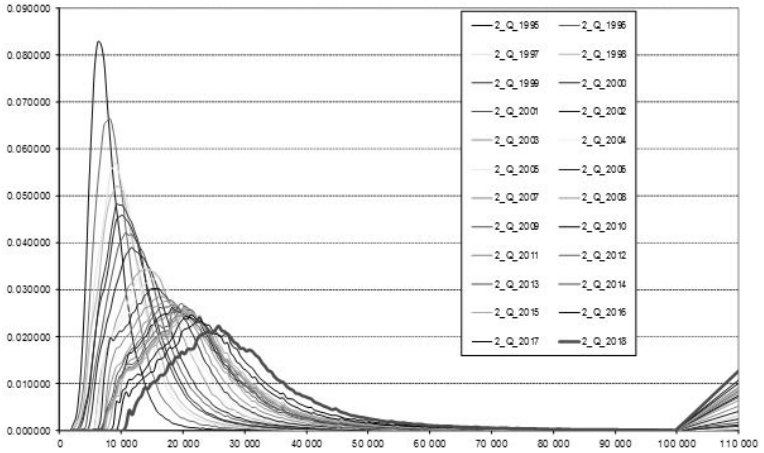

Figure 2: Wage distributions in CR 1995-2018.

smooth over time, mixture models, in particular normal mixture, give good results now and are more promising for the future [4].

The probability density for a general model of a normal mixture can be written as

$$f(x) = \sum_{i=1}^{n} p_i g_i(x) \tag{1}$$

where $g_i(x)$ is the probability density of normal distribution

$$g_i(x) = \frac{1}{\lambda_i \sqrt{2\pi}} exp\left(-\frac{(x-\theta_i)^2}{2\lambda_i^2}\right), \tag{2}$$

n is the number of components in the mixture and p is the vector of weights, for which

$$0 < p_i < 1, \forall i, \sum_{i=1}^{n} p_i = 1. \tag{3}$$

When considering a mixture model, the key question is what should be the components of the model. A mixture model can be created using two different approaches. In the first approach, the model is created manually, based on user-specified components, their weights and parameters of the components. In the second approach, the model is created completely from data. The partitioning will be done automatically, we need not even to specify the number of components.

The first approach, in general, resembles the analytical (or explanation-based) approach to machine learning, where some initial domain knowledge is refined using only a few data. Here, we can use expert knowledge about known factors that

influence the wage to define the components in the mixture. We will refer to this approach as "Scenario 1" later on. The second approach follows the empirical learning paradigm, where we are using only the data to build a model. We can distinguish two basic frameworks of empirical learning: learning as approximation and learning as search. In the former case, we restrict ourself to pre-defined class of models (e.g. linear functions in the case of linear regression or given number of clusters for the k-means clustering algorithm) and use the data to find best parameters "only" within this class. In the later case, we search the whole space of all possible models to find the best one, so we do not specify the type of function when creating a regression model or do not specify the number of resulting clusters as is the case for hierarchical cluster analysis. To follow the learning as approximation framework in our experiments, we will specify the number of components (which we derive from the Scenario 1) but not the components themselves - this will be the Scenario 2 of our experiments. To follow the learning as search framework, we will specify (in Scenario 3) only the maximal number of the components (to reduce the space of the models to be searched).

4 Experiments and Results

For modeling purposes we used the data only for the period 2000-2018 and further bound the empirical wage distributions by 100 000 CZK (ignoring the heavy tail) as there were very few employees with wages above this value in the data. We consider mixtures of up to 6 components in our experiments.

To assess the quality of the models, we used the Akaike Information Criterion (AIC) [1]. Given a collection of models for given data, AIC estimates the quality of each model, relative to each of the other models; the lower is the value the better is the model. AIC deals with the trade-off between the goodness-of-fit of the model and the simplicity of the model. The goodness-of-fit is expressed using the likelihood of the model (probability of observed data given the model), the simplicity of the model is expressed using the number of parameters of the model. The basic form of AIC is given in formula 4. Here k is the number of the parameters of the model.

$$AIC = -2LogLikelihood + 2k \tag{4}$$

We used a corrected AIC which considers also the number of observations n. This version of the criterion is shown in formula 5.

$$AICc = -2LogLikelihood + 2k + \frac{2k(k+1)}{n-k-1} \tag{5}$$

We used Jmp, a data analysis tool developed by SAS (http://www.jmp.com) for our computations. The system looks for maximum likelihood estimates so using AICc to compare the models is very straightforward.

4.1 Scenario 1

A number of factors influence the economic activities and the wages. Among them, the most important is the gender followed by age and education (see e.g. [6, 2]). So we used these factors to define the components of the mixture. We considered the (obvious) two categories of gender (female, male), three categories of age (below 30, 30-50, above 50) and six categories of education (basic, apprenticeship, secondary, bachelor, master, PhD). We created three different mixture models based on the above mentioned factors, i.e. a 2-component model based on gender, a 3-component model based on age and a 6-component model based on education for every year from the period 2000-2018. Figures 3, 4 and 5 show the weights of the components and the average wages. The figures also show the comparison of average wages within the categories with the average wage in the whole CR (displayed as dots in the right graphs). While the average wage in the whole CR seems to be "in the middle" between average wage of men and average wage of women, the average wage in the whole CR corresponds to the average wage of people above 50 as well as to the average wage of people with secondary education. It can be also seen that the weights of the components gender and age are almost stable over time and the components education exhibit linear trend.

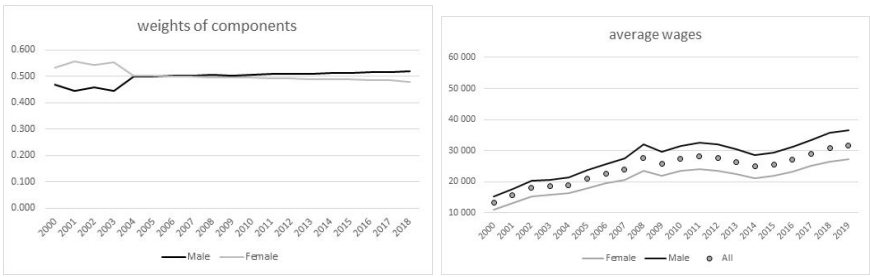

Figure 3: Components defined by gender. Weights of components (left), average wages (right).

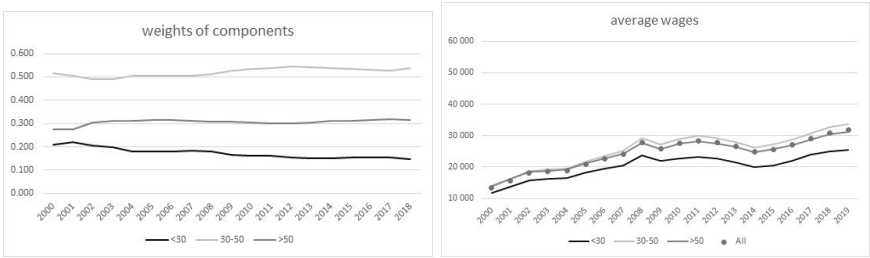

Figure 4: Components defined by age. Weights of components (left), average wages (right).

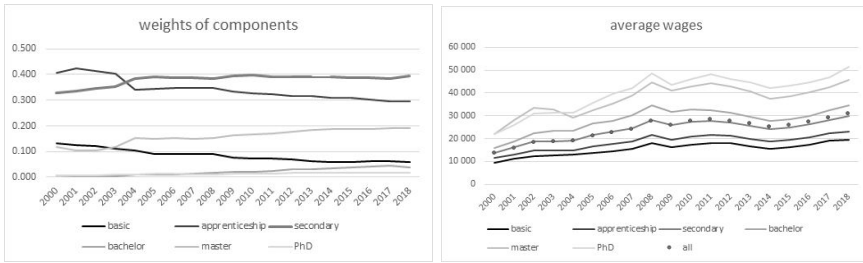

Figure 5: Components defined by education. Weights of components (left), average wages (right).

4.2 Scenario 2

Like in Scenario 1, we again created three models (2-component model, 3-component model and 6-component model) for every year from the period 2000-2018. But now we didn't specified the components to be used. Figures 6, 7 and 8 again show the weights of the components and the average wages for the corresponding models. With increasing number of components the average wages become more or less constant, so the changes of the wage distributions over time are modeled by modifying the weights of the components. Unlike scenario 1, there is no clear relationship between the average wages in the whole CR and average wages for a component.

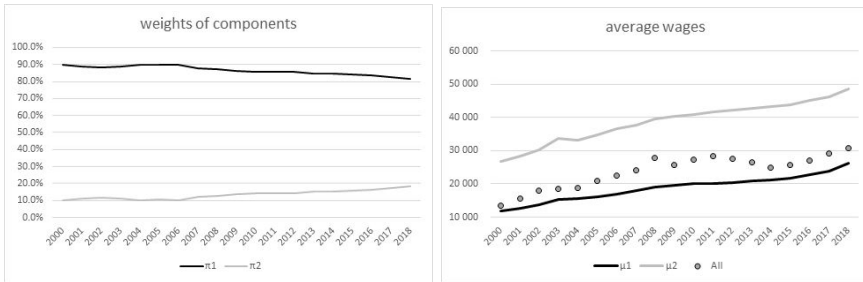

Figure 6: Model with 2 components. Weights of components (left), average wages (right).

4.3 Scenario 3

We aimed at creating a single model with up to 6 components for every year from the period 2000-2018. As the used software Jmp does not support this scenario directly, we created a model with different number of components (2 to 6) and choose the model with the best (lowest) value of AICc criterion for every year. It turned out that the model with highest number of components was always the best

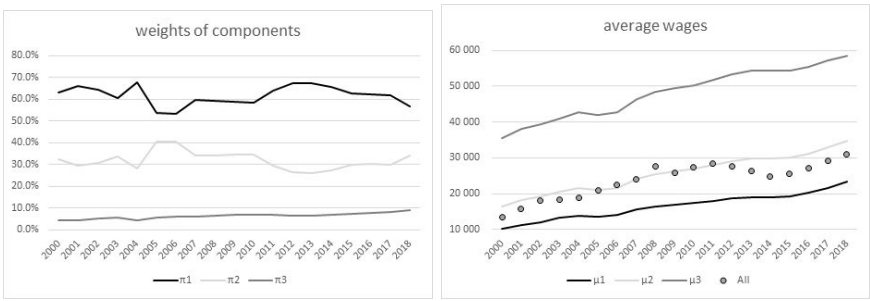

Figure 7: Model with 3 components. Weights of components (left), average wages (right).

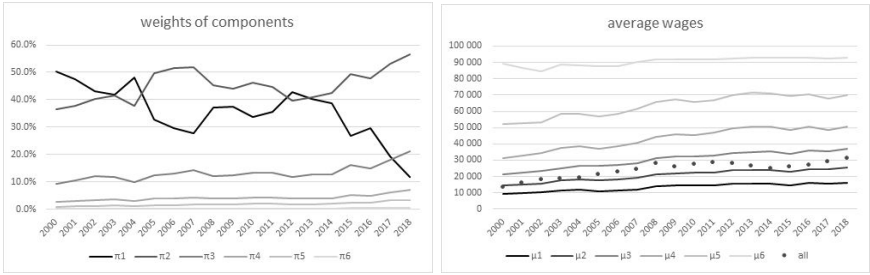

Figure 8: Model with 6 components. Weights of components (left), average wages (right).

one. And it seems that the more components in the mixture model, the better the model will be. This is because increasing the number of parameters in the model almost always improves the value of AICc. We confirmed this by creating an extra model with 10 components (see figure 9).

5 Discussion

The experiments reported in section 4 confirmed our expectation concerning interpretability of the models. Obviously, only models where the components were determined in advance using some socio-demographic characteristics can be fully understandable. Concerning the models, where the components were created automatically, we can only guess their meaning. So in case of 2-component model presented in Fig. 6 we might think that the 2 components correspond to people with high and low wage. The experiments also confirmed our expectation concerning stability. The models created according to the scenario 2 and 3 are less stable (in the sense of more significant changes of weights of components over time) than models created according to the scenario 1. This instability increases with

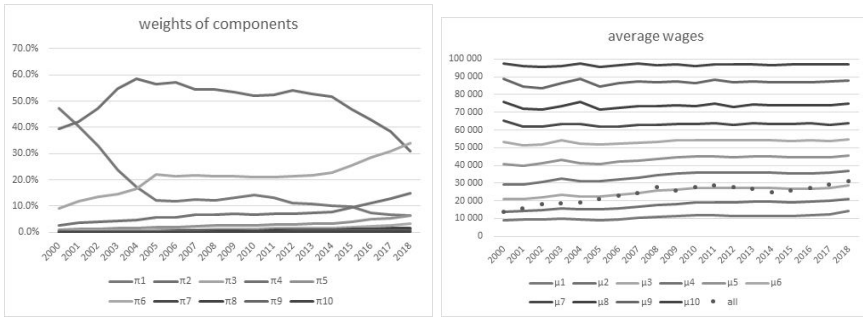

Figure 9: Model with 10 components. Weights of components (left), average wages (right).

the number of components, this might be the effect of the fact that the average wages within the components became stable (constant) and thus different wage distributions as they change over time are modeled mainly by different weights of the components.

The quality of the models in terms of AICc is summarized in tables shown in figures 10 and 11. The values of this criterion are very high, probably due to the large size of the data sample. As stated in section 4, lower values of AICc means better model. But having values of AICc in the order of tens of millions, the differences between values of AICc shown in the tables are rather small; the largest relative difference between best and worst value of AICc is 9,6% (the best value of AICc corresponds to 100%). Within a single scenario when increasing the number of components we always obtained a model with better value of AICc. So using scenario 3, we will end up with the most complex model that uses the upper bound for the number of components (the value of AICc for this model is displayed in bold). When comparing models with the same number of components between scenario 1 and scenario 2, we can see that in most cases the model created according to scenario 1 has better values of AICc than the corresponding model created according to scenario 2. For mixture models with 3 and 6 components, the model created according to scenario 1 has always better values of AICc than model created according to scenario 2, for mixture models with 2 components, the model created according to scenario 1 has better values of AICc in about half times than model created according to scenario 2 - the better value of AICc is displayed in italics. This was a rather unexpected result as we assumed a trade-off between lower interpretability and higher accuracy of models created according to scenario 2 compared to models created according to scenario 1. We also found out, that for models created according to scenario 1, more components do not assure model with better values of AICc (in about half cases, the best model created using scenario 1 was the model using age not the model using education to define three components - the lowest AICc value is displayed in bold).

The applicability of the models for prediction is better for models created under scenario 1. This is because the trends of average wages in "natural" segments (components) as well as the trends of proportions of these segments in the whole population can be better predicted than the average wages and proportions of unclearly defined groups of people.

	2comp		3comp		6comp		
year	scenario1	scenario2	scenario1	scenario2	scenario1	scenario2	rel. diff.
2000	22 195 838	*21 093 074*	**20 252 927**	21 009 184	*20 388 764*	20 984 567	9.6%
2001	*21 680 105*	21 682 984	*21 365 275*	21 603 105	**21 181 509**	21 576 242	2.4%
2002	*22 169 171*	22 472 272	**21 435 866**	22 397 684	*21 924 464*	22 365 388	4.8%
2003	*24 937 067*	25 133 022	**24 871 565**	25 066 600	*24 953 605*	25 032 183	1.1%
2004	*33 834 474*	34 305 405	**33 339 583**	34 221 075	*34 094 250*	34 184 645	2.9%
2005	37 529 873	*37 371 261*	**36 767 347**	37 265 424	*36 985 957*	37 208 142	2.1%
2006	*37 367 892*	40 732 664	*40 323 597*	40 634 385	**40 244 699**	40 562 494	9.0%
2007	41 001 395	*40 825 108*	**39 791 822**	40 729 070	*40 404 797*	40 655 321	3.0%
2008	41 931 336	*41 645 965*	*41 477 166*	41 554 579	**40 878 209**	41 510 467	2.6%
2009	41 496 006	*40 376 914*	*39 807 479*	40 287 403	**39 632 051**	40 247 314	4.7%
2010	41 446 196	*40 928 365*	*40 532 806*	40 836 614	**40 486 768**	40 792 348	2.4%
2011	41 927 106	*41 298 305*	*40 428 498*	41 222 303	**40 265 343**	41 181 388	4.1%
2012	42 531 004	*41 901 055*	*41 088 433*	41 829 204	**40 817 360**	41 801 444	4.2%
2013	42 667 162	*42 337 079*	**41 647 277**	42 267 984	*41 767 306*	42 239 549	2.4%
2014	43 588 667	*43 248 790*	**42 439 401**	43 177 231	*42 821 466*	43 147 021	2.7%
2015	*43 807 582*	44 215 338	*43 565 302*	44 139 818	**43 319 251**	44 090 949	2.1%
2016	*44 562 415*	44 718 118	*44 393 742*	44 645 288	**44 330 239**	44 610 307	0.9%
2017	*45 809 268*	46 226 281	**45 243 759**	46 152 686	*46 026 411*	46 105 397	2.2%
2018	*46 755 230*	47 465 278	**46 460 716**	47 395 327	*46 686 638*	47 343 505	2.2%

Figure 10: AICc for scenario 1 and 2.

6 Conclusions

The aim of our experiments was to create mixture models of wage distribution using different amount of domain knowledge. We propose three scenarios that were inspired by analytical learning, learning as approximation and learning as search approaches. When specifying the components of the mixture in advance in terms of socio-demographic characteristics, we will obtain easily interpretable models. When specifying only the number of components or the maximal number of components, the mixture models become less interpretable as the components do not have clear meaning in terms of natural groups of people. We expected this result but we did not expect that the models created using "natural" components will be slightly better (in terms of lower values of AICc). So our results contradict the idea to create models of wage distributions solely from data as proposed e.g. in [3].

year	2comp	3comp	4comp	5comp	6comp	...	*10comp*
2000	21 093 074	21 009 184	20 988 862	20 984 871	**20 984 567**	...	***20 983 186***
2001	21 682 984	21 603 105	21 581 980	21 576 781	**21 576 242**		***21 572 648***
2002	22 472 272	22 397 684	22 372 055	22 366 180	**22 365 388**		***22 360 529***
2003	25 133 022	25 066 600	25 038 439	25 033 681	**25 032 183**		***25 022 382***
2004	34 305 405	34 221 075	34 194 124	34 187 776	**34 184 645**		***34 154 753***
2005	37 371 261	37 265 424	37 222 611	37 213 065	**37 208 142**		***37 175 084***
2006	40 732 664	40 634 385	40 579 976	37 213 065	**40 562 494**		***40 532 481***
2007	40 825 108	40 729 070	40 673 081	40 664 204	**40 655 321**		***40 632 827***
2008	41 645 965	41 554 579	41 523 341	41 516 741	**41 510 467**		***41 486 717***
2009	40 376 914	40 287 403	40 259 437	40 253 304	**40 247 314**		***40 227 188***
2010	40 928 365	40 836 614	40 809 884	40 802 540	**40 792 348**		***40 775 896***
2011	41 298 305	41 222 303	41 199 262	41 191 830	**41 181 388**		***41 162 295***
2012	41 901 055	41 829 204	41 817 073	41 811 720	**41 801 444**		***41 765 392***
2013	42 337 079	42 267 984	42 256 562	42 250 703	**42 239 549**		***42 200 054***
2014	43 248 790	43 177 231	43 163 881	43 159 010	**43 147 021**		***43 103 946***
2015	44 215 338	44 139 818	44 124 144	44 116 019	**44 090 949**		***44 059 768***
2016	44 718 118	44 645 288	44 631 413	44 624 494	**44 610 307**		***44 571 404***
2017	46 226 281	46 152 686	46 141 536	46 131 456	**46 105 397**		***46 080 693***
2018	47 465 278	47 395 327	47 380 948	47 362 874	**47 343 505**	...	***47 331 211***

Figure 11: AICc for scenario 3.

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Questions Concerning Composition of Discrete Multivariate Uncertain Measures in Liu's Uncertainty Theory

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Abstract

An Uncertainty Theory founded by Baoding Liu in 2007 claims to be a universal tool for a correct description of uncertainty aiming to overcome the limitations of probability theory and other alternative approaches. Since the dependency structures of variables in the last decades appeared to be an essential and fruitful field of research and an important issue for the decision-making problems, a fundamental question arise whether the Liu's Uncertainty Theory is able to describe the complex character of dependency structures among the set of uncertain variables. The presented paper aims to contribute to this field, attempts to define an operator of composition serving as a tool for building multivariate uncertain measures and as a result opens several unanswered questions.

1 Introduction

The presented paper in a way extends the authors' acquaintance with Baoding Liu's Uncertainty Theory [7] and its usability for modeling of multivariate distributions of discrete quantities (see Bína and Váchová [1]) where we endeavored to propose a way to build up multivariate models using an operator of composition defined in a similar way to other theoretical approaches as probability theory (see definition of operator of composition and its properties, e.g. in a fundamental paper by Jiroušek [4]), in a general framework of probabilistic models (see Jiroušek

and Shenoy [3]), in valuation-based systems (see Jiroušek and Shenoy [2]) or in Dempster–Shafer Evidence Theory (see Jiroušek and Vejnarová [5]).

In our 2015 Czech–Japan Seminar paper [1] we claimed that more profound analysis of composition approach under the framework of Uncertainty Theory is necessary and in this paper we make an effort to discuss possible definitions of composition together with the philosophical fundamentals and other important properties of Liu's Uncertainty Theory. Meanwhile, the field of uncertainty theory has earned many citations and application (from the vast field let us mention, e.g., Zeng et al. [9]). Thus, this paper comprises another step in assessing the Uncertainty Theory and its usability as a possible tool for modeling using multivariate distributions.

2 Fundamentals of Uncertainty Theory

The fundamentals of Liu's Uncertainty Theory [7] stand on two basic principles and a set of axioms. The two principles are:

Law of Truth Conservation Similarly as some other descriptions of uncertainty Baoding Liu does not fully agree with the well-known "tertium non datur" or the "law of excluded third" (saying that the proposition must be either true or false) and the law of non-contradiction (contradictory propositions cannot be both true in the same sense and time). Instead, Liu proposes the law of truth conservation, stating that *The truth values of a proposition and its negation should sum to unity.*

Maximum Uncertainty Principle It is probably intuitive that there is no uncertainty if we are sure that an uncertain measure of an event takes one of its extreme values, i.e., it is equal to 0 or 1. Similarly, the uncertain measure equal to 0.5 stands for the maximum uncertainty because some event and its complement have equal "likelihood" of occurrence. In agreement with this, Baoding Liu postulates a maximum uncertainty principle saying that: *For any event, if there are multiple reasonable values that an uncertain measure may take, then the value as close to 0.5 as possible is assigned to the event.*

The set of axioms in Liu's Uncertainty Theory is obviously and quite naturally inspired by the famous Kolmogorov's axiomatization of probability theory. First of all, we will clarify and define the basic notions and notation. Let Γ be a nonempty set called the *universal set*. An *algebra* $\mathcal{L}$ is a collection of subsets from Γ such that Γ is an element of this collection and $\mathcal{L}$ is closed under complementation (with respect to Γ) and finite union. The collection $\mathcal{L}$ is called *σ-algebra* if it is closed under countable union. Having a nonempty universal set Γ, collection $\mathcal{L}$ and a σ-algebra over Γ we call the ordered pair $(\Gamma, \mathcal{L})$ a *measurable space* and any element Λ of $\mathcal{L}$ is called a *measurable set* or an *event*.

An *uncertain measure* $\mathcal{M}$ on the σ-algebra $\mathcal{L}$ assigns a number $\mathcal{M}\{\Lambda\}$ to each event Λ representing the belief degree (not frequency) expressing the strength of

trust that Λ will occur. Naturally, this assignment must fulfill the following properties summarized by Baoding Liu in the following set of axioms.

Axiom 1 (Normality Axiom). $\mathcal{M}\{\Gamma\} = 1$ *for the universal set* Γ.

Axiom 2 (Duality Axiom). $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ *and its complement (with respect to* Γ*)* Λ^c.

Axiom 3 (Subadditivity Axiom). *For every countable sequence of events* Λ_1, Λ_2, *... we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The duality axiom is just an application of the law of truth conservation. Liu claims that there does not exist a general formula allowing to evaluate the belief degree for the union of events using the belief degrees of individual events. Neither the probabilistic additivity axiom nor possibilistic maximum works. He claims that perhaps there is no stronger rule than the subadditivity.

From the above presented set of axioms we can infer several interesting properties (for deeper understanding see [7]).

Theorem 1 (Uncertain Measure of Empty Set). *Having an uncertain measure* $\mathcal{M}$ *it holds that*

$$\mathcal{M}\{\emptyset\} = 0.$$

Proof. Starting from universal set Γ, normality axiom says $\mathcal{M}\{\Gamma\} = 1$. Since $\Gamma^c = \emptyset$, from duality axiom we get

$$\mathcal{M}\{\emptyset\} = 1 - \mathcal{M}\{\Gamma\} = 0.$$

□

Theorem 2 (Monotonicity Theorem). *Uncertain measure* $\mathcal{M}$ *is an increasing set function, i.e., for* $\Lambda_1 \subset \Lambda_2$ *holds*

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}.$$

Proof. Again we use normality axiom which says that $\mathcal{M}\{\Gamma\} = 1$ and from duality axiom we have

$$\mathcal{M}\{\Lambda_1^c\} = 1 - \mathcal{M}\{\Lambda_1\}.$$

Since $\Lambda_1 \subset \Lambda_2$ we can express the universal set as $\Gamma = \Lambda_1^c \cup \Lambda_2$ and using the subadditivity axiom we get

$$1 = \mathcal{M}\{\Gamma\} \leq \mathcal{M}\{\Lambda_1^c\} + \mathcal{M}\{\Lambda_2\} = 1 - \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\}.$$

And therefore $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$. □

Now, from normality axiom, zero uncertain measure of empty set and from monotonicity it follows that for an uncertain measure $\mathcal{M}$ and any event Λ it holds that

$$0 \leq \mathcal{M}\{\Lambda\} \leq 1.$$

For nonempty universal set Γ, σ-algebra $\mathcal{L}$ over Γ and uncertain meansure $\mathcal{M}$ the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an *uncertainty space*.

Uncertain variable is now defined analogously to a probabilistic random variable.

Definition 1 (Uncertain Variable)**.** *An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B of real numbers.*

3 Product in Uncertainty Theory

To introduce a product of two independent measures we can introduce a product uncertain measure. For uncertainty spaces $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ for $k = 1, 2, \dots$ we can define a *product universal set* as a cartesian product

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots,$$

i.e., the set of all order tuples generated using the considered universal sets in the form $(\gamma_1, \gamma_2, \dots)$, where $\gamma_k \in \Gamma_k$ for $k = 1, 2, \dots$. Now *measurable rectangle* in Γ is a cartesian product

$$\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots$$

where Λ_k *in*$\mathcal{L}_k$ for $k = 1, 2, \dots$ The smallest σ-algebra containing all measurable rectangles in Γ is called the *product σ-algebra*

$$\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots.$$

Then the *product uncertain measure* $\mathcal{M}$ on the product σ-algebra $\mathcal{L}$ is introduced in the following axiom (some variant was introduced in [6]).

Axiom 4 (Product Axiom I)**.** *Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \dots$, then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \dots$.

Remark 1. Let us denote that it is rather surprising to introduce the product uncertain measure in the form of axiom. As far as we know, the product measure is usually introduced as a definition (e.g., in probability theory).

In his paper published in Journal of Uncertain Systems [6] Liu states that the product axiom can be introduced also in some other way and proposes a "usual" product employing multiplication instead of minimum.

Axiom 5 (Product Axiom II). *Let* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ *be uncertainty spaces for* $k = 1, 2, \ldots$, *then the product uncertain measure* $\mathcal{M}$ *is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \prod_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k *are arbitrarily chosen events from* $\mathcal{L}_k$ *for* $k = 1, 2, \ldots$.

Let us also notice that the product axiom in this paper introduces product uncertain measure only for rectangles. For possible generalization, see again Liu [7].

3.1 Examples on Product

Let us start with a pair of uncertain variables ξ and ψ both dichotomic with uncertain measure given for all events by Table 1.

Table 1: Values of Uncertain Measure for Uncertain Variables ξ and ψ.

Uncertain Variable ξ	
Event	**Un. M.**
{}	0
{N}	0.3
{O}	0.7
{N,O}	1

Uncertain Variable ψ	
Event	**Un. M.**
{}	0
{C}	0.6
{E}	0.4
{C,E}	1

For the pair of uncertain variables ξ and ψ given by Table 1 we can easily compute their product uncertain measure in "minimum variant", i.e., using Axiom 4.

Table 2: Product uncertain measure of uncertain variables ξ and ψ according to the Axiom 4, i.e. product variant based on minimum.

$\mathcal{M}_\wedge\{\xi,\psi\}$		ψ			
		{}	{C}	{E}	{C,E}
ξ	{}	0	0	0	0
	{N}	0	0.3	0.3	0.3
	{O}	0	0.6	0.4	0.7
	{N,O}	0	0.6	0.4	1

The exact proof that the resulting product uncertain measure based on minimum fulfills set of Axiom 1 – 3 can be found in Peng and Iwamura [8].

Notice that the last column of Table 2 contains the uncertain measure of uncertain variable ξ and the last row is the uncertain measure of uncertain variable ψ which means that the "marginal distributions" are instantly available. Let us remark that this joint table is a product uncertain measure of two independent

variables. The requirement of monotonicity implies that in case of dependent variables, some (or all) of the "central" four numbers can be only smaller. An example of dependent variables is given in Table 3.

Table 3: Uncertain measure of dependent uncertain variables ξ and ψ.

$\mathcal{M}_{\mathrm{dep}}\{\xi,\psi\}$		ψ			
		{}	{C}	{E}	{C,E}
ξ	{}	0	0	0	0
	{N}	0	0.1	0.3	0.3
	{O}	0	0.6	0.2	0.7
	{N,O}	0	0.6	0.4	1

Where lower bounds are constrained also according to the requirement of subadditivity in the following sense (the lower bounds for "central" cells were added in such a way that the sum of "central" cell gives the respective "marginal" or the uncertain measure of compound event). For this simple example the following Table 4 provides constraints for the "central" values.

Table 4: Constraints of two-dimensional uncertain measure of variables ξ and ψ.

$\mathcal{M}_{\mathrm{bounds}}\{\xi,\psi\}$		ψ			
		{}	{C}	{E}	{C,E}
ξ	{}	0	0	0	0
	{N}	0	0-0.3	0-0.3	0.3
	{O}	0	0.3-0.6	0.1-0.4	0.7
	{N,O}	0	0.6	0.4	1

As we already mentioned, one other possible form of product uncertain measure can be defined according to the Axiom 5 (see Liu in JUS [6]). Again an example is given in the following Table 5.

Table 5: Product uncertain measure of uncertain variables ξ and ψ according to the Axiom 5, i.e. product variant based on multiplication.

$\mathcal{M}_{\times}\{\xi,\psi\}$		ψ			
		{}	{C}	{E}	{C,E}
ξ	{}	0	0	0	0
	{N}	0	0.18	0.12	0.3
	{O}	0	0.42	0.28	0.7
	{N,O}	0	0.6	0.4	1

We can see that the second variant of product axiom formulation (Axiom 5 based on multiplication) provide results in agreement with the set of fundamental Axioms 1–3. But the use of multiplication leads to a deviation from the basic Principle of maximal uncertainty since the result contains lower (i.e. more "certain") values which in the most case are farther from 0.5 than the product based on minimum and formulated in the first variant (Axiom 4). Though we did not succeed in finding original Liu's the argumentation, this seems to be a reason why the variant

of Axiom 5, i.e., product based on multiplication, is no longer mentioned in more recent texts of Baoding Liu (e.g., in [7]).

4 Attempt to define an operator of composition

In the introduction, we mentioned few papers where the operator of composition as an important mean of aggregation of multivariate distributions was defined. The kind reader can see that the composition can be formulated either using a product of a (multivariate) distribution with a conditional distribution (conditioned by variables contained in both distributions) or – under the framework of probability theory – equivalently using a product of two (multivariate) distributions divided by marginal of the second distribution in variables appearing again in both distributions. This provides the first ideas for the possible definitions of composition for two multivariate uncertain measures.

In this section, the operation of composition will be documented on an example of two bivariate discrete uncertain measures defined by the following Tables 6 and 7 (where the second table was used above as an example Table 3).

Table 6: Bivariate uncertain measure of two uncertain variables ϕ and ξ.

$\mathcal{M}_1\{\phi,\xi\}$		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.07	0.06	0.1
	{Y}	0	0.1	0.3	0.4
	{G}	0	0.2	0.5	0.6
	{R,Y}	0	0.1	0.35	0.4
	{R,G}	0	0.2	0.5	0.6
	{Y,G}	0	0.25	0.65	0.9
	{R,Y,G}	0	0.3	0.7	1

Table 7: Bivariate uncertain measure of two uncertain variables ξ and ψ.

$\mathcal{M}_2\{\xi,\psi\}$		ψ			
		{}	{C}	{E}	{C,E}
ξ	{}	0	0	0	0
	{N}	0	0.1	0.3	0.3
	{O}	0	0.6	0.2	0.7
	{N,O}	0	0.6	0.4	1

4.1 Composition Based on: Conditional Uncertain Measure

The first considered possibility will be an attempt to define the operator of composition using a product of multivariate uncertain measure with a conditional uncertain measure.

A *conditional uncertain measure* of an event A given some fixed event B, is in agreement with the given set of axioms and maximum uncertainty principle defined (again by Liu [7]) it in the following way.

Definition 2 (Conditional Uncertain Measure)**.** *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $A, B \in \mathcal{L}$. Then for all $\mathcal{M}\{B\} > 0$ the conditional uncertain measure of A given B is defined by*

$$\mathcal{M}\{A|B\} = \begin{cases} \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} & \text{if} \quad \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} < 0.5, \\ 1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} & \text{if} \quad \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} < 0.5, \\ 0.5 & \text{otherwise.} \end{cases}$$

It can be easily shown that conditional uncertain measure $\mathcal{M}\{A|B\}$ is an uncertain measure and $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ is an uncertainty space.

Now, let us use a Definition 2 and compute a conditional uncertain measure $\mathcal{M}\{\psi|\xi\}$. The result is in teh following Table 8.

Table 8: Bivariate conditional uncertain measure $\mathcal{M}\{\psi|\xi\}$

$\mathcal{M}\{\psi\|\xi\}$		ψ			
		{}	{C}	{E}	{C,E}
ξ	{}	0	0	0	0
	{N}	0	1/3	2/3	0.3
	{O}	0	5/7	2/7	0.7
	{N,O}	0	0.6	0.4	1

Let us denote that all rows (as conditional uncertain measures) must fulfill the duality property. To ensure this, in case of values in the second row and third column (and also in the third row and second column) the second variant in a definition of the conditional uncertain measure was used.

Now, we attempt to define an operator of composition using the conditional uncertain measure. Let us have again uncertainty spaces $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ for $k \in K$ and their cartesian product $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots$, i.e., the set of all order tuples generated using the considered universal sets in the form $(\gamma_1, \gamma_2, \dots)$, where $\gamma_k \in \Gamma_k$ for $k \in K$. Measurable rectangle in Γ is a cartesian product $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots$ where $\Lambda_k \in \mathcal{L}_k$ for $k \in K$ The smallest σ-algebra containing all measurable rectangles in Γ is so called the product σ-algebra $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots$

Definition 3 (Operator of Composition – Based on Conditional Uncertain Measure)**.** *Let $L, M \subset K$ such that $L \cup M = K$ and induce a pair of uncertainty spaces $(\Gamma_L, \mathcal{L}_L, \mathcal{M}_L)$ and $(\Gamma_M, \mathcal{L}_M, \mathcal{M}_M)$. Composition of measures $\mathcal{M}_L$ a $\mathcal{M}_M$ is defined by*

$$\mathcal{M}_L\{\Lambda_L\} \triangleright \mathcal{M}_M\{\Lambda_M\} = \mathcal{M}_L\{\Lambda_L\} \cdot \mathcal{M}_M\{\Lambda_{M \setminus L} | \Lambda_{L \cap M}\}$$

where $\Lambda_{M \setminus L}$ and $\Lambda_{L \cap M}$ are events from the respective uncertainty spaces.

Table 9: Result of a composition of two two-dimensional uncertain measures based on a conditional distribution.

ψ:{}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0	0	0
	{Y}	0	0	0	0
	{G}	0	0	0	0
	{R,Y}	0	0	0	0
	{R,G}	0	0	0	0
	{Y,G}	0	0	0	0
	{R,Y,G}	0	0	0	0

ψ:{C}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.07	0.06	0.1
	{Y}	0	0.1	0.3	0.4
	{G}	0	0.2	0.5	0.6
	{R,Y}	0	0.1	0.35	0.4
	{R,G}	0	0.2	0.5	0.6
	{Y,G}	0	0.25	<u>0.65</u>	0.6
	{R,Y,G}	0	0.3	<u>0.7</u>	0.6

ψ:{E}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.07	0.06	0.1
	{Y}	0	0.1	2/7	0.4
	{G}	0	0.2	2/7	0.4
	{R,Y}	0	0.1	2/7	0.4
	{R,G}	0	0.2	2/7	0.4
	{Y,G}	0	0.25	2/7	0.4
	{R,Y,G}	0	0.3	2/7	0.4

ψ:{C,E}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.07	0.06	0.1
	{Y}	0	0.1	0.3	0.4
	{G}	0	0.2	0.5	0.6
	{R,Y}	0	0.1	0.35	0.4
	{R,G}	0	0.2	0.5	0.6
	{Y,G}	0	0.25	0.65	0.9
	{R,Y,G}	0	0.3	0.7	1

Using this definition a result of composition $\mathcal{M}_1\{\phi,\xi\} \triangleright \mathcal{M}_2\{\xi,\psi\} = \mathcal{M}_1\{\phi,\xi\} \cdot \mathcal{M}\{\psi|\xi\}$ is not a multivariate uncertain measure, since it violates the property of monotonicity (see underlined elements in an example of composition in Table 9).

4.2 Composition Based on: Product with Multiplication

Another rather intuitive variant is to take as an inspiration the second variant of product axiom (an Axiom 5 based on multiplication) and to divide the product of two multivariate distributions by the marginal of the second one containing all common variables.

Definition 4 (Operator of Composition – Based on Product Axiom in Multiplication Version)**.** *Let $L, M \subset K$ such that $L \cup M = K$ and induce a pair of uncertainty spaces $(\Gamma_L, \mathcal{L}_L, \mathcal{M}_L)$ and $(\Gamma_M, \mathcal{L}_M, \mathcal{M}_M)$. Composition of measures $\mathcal{M}_L$ a $\mathcal{M}_M$ is defined by*

$$\mathcal{M}_L\{\Lambda_L\} \triangleright \mathcal{M}_M\{\Lambda_M\} = \frac{\mathcal{M}_L\{\Lambda_L\} \cdot \mathcal{M}_M\{\Lambda_M\}}{\mathcal{M}_M\{\Lambda_{L\cap M}\}}$$

where $\Lambda_{L\cap M}$ is an event from the respective uncertainty space.

Again, using this definition a result of composition

$$\mathcal{M}_1\{\phi,\xi\} \triangleright \mathcal{M}_2\{\xi,\psi\} = \frac{\mathcal{M}_1\{\phi,\xi\} \cdot \mathcal{M}_2\{\xi,\psi\}}{\mathcal{M}_2\{\xi\}}$$

Table 10: Result of composition of two two-dimensional uncertain measures based on a multiplication.

ψ:{}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0	0	0
	{Y}	0	0	0	0
	{G}	0	0	0	0
	{R,Y}	0	0	0	0
	{R,G}	0	0	0	0
	{Y,G}	0	0	0	0
	{R,Y,G}	0	0	0	0

ψ:{C}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.023	0.051	0.06
	{Y}	0	0.033	<u>0.257</u>	0.24
	{G}	0	0.067	<u>0.429</u>	0.36
	{R,Y}	0	0.033	<u>0.3</u>	0.24
	{R,G}	0	0.067	<u>0.429</u>	0.36
	{Y,G}	0	0.083	<u>0.557</u>	0.54
	{R,Y,G}	0	0.1	0.6	0.6

ψ:{E}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.07	0.017	0.04
	{Y}	0	0.1	0.086	0.16
	{G}	0	0.2	0.143	0.24
	{R,Y}	0	0.1	0.1	0.16
	{R,G}	0	0.2	0.143	0.24
	{Y,G}	0	0.25	0.186	0.36
	{R,Y,G}	0	0.3	0.2	0.4

ψ:{C,E}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.07	0.06	0.1
	{Y}	0	0.1	0.3	0.4
	{G}	0	0.2	0.5	0.6
	{R,Y}	0	0.1	0.35	0.4
	{R,G}	0	0.2	0.5	0.6
	{Y,G}	0	0.25	0.65	0.9
	{R,Y,G}	0	0.3	0.7	1

is not a multivariate uncertain measure, since also in this case it violates the property of monotonicity (see again the underlined elements in an example of composition in Table 10).

4.3 Composition based on: Product with Minimum

We saw that the previous possibilities together with the definition given in preceding paper [1] all lead to the violation of the set of axioms (particularly the requirement of monotonicity). Therefore, the last attempt efforts to stand on the philosophical basis of Liu's Uncertainty Theory. This variant of the definition of the operator of composition will be formulated simply as a product in its multiplication variant.

Definition 5 (Operator of Composition – Based on Product Axiom in Minimum Version)**.** *Let $L, M \subset K$ such that $L \cup M = K$ and induce a pair of uncertainty spaces $(\Gamma_L, \mathcal{L}_L, \mathcal{M}_L)$ and $(\Gamma_M, \mathcal{L}_M, \mathcal{M}_M)$. Composition of measures $\mathcal{M}_L$ a $\mathcal{M}_M$ is defined by*

$$\mathcal{M}_L\{\Lambda_L\} \triangleright \mathcal{M}_M\{\Lambda_M\} = \min\left(\mathcal{M}_L\{\Lambda_L\}, \mathcal{M}_M\{\Lambda_M\}\right).$$

Now, the result of composition in our example of pair of bivariate uncertain measures $\mathcal{M}_1\{\phi, \xi\} \triangleright \mathcal{M}_2\{\xi, \psi\} = \min\left(\mathcal{M}_1\{\phi, \xi\} \triangleright \mathcal{M}_2\{\xi, \psi\}\right)$ is summarized in the Table 11.

According to our current insight, this variant is the only one "clean" in a philosophical sense. It does not violate the requirement of monotonicity and other

Table 11: Result of composition of two two-dimensional uncertain measures based on a minimum.

ψ:{}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0	0	0
	{Y}	0	0	0	0
	{G}	0	0	0	0
	{R,Y}	0	0	0	0
	{R,G}	0	0	0	0
	{Y,G}	0	0	0	0
	{R,Y,G}	0	0	0	0

ψ:{C}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.07	0.06	0.1
	{Y}	0	0.1	0.3	0.4
	{G}	0	0.1	0.5	0.6
	{R,Y}	0	0.1	0.35	0.4
	{R,G}	0	0.1	0.5	0.6
	{Y,G}	0	0.1	0.6	0.6
	{R,Y,G}	0	0.1	0.6	0.6

ψ:{E}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.07	0.06	0.1
	{Y}	0	0.1	0.2	0.4
	{G}	0	0.2	0.2	0.4
	{R,Y}	0	0.1	0.2	0.4
	{R,G}	0	0.2	0.2	0.4
	{Y,G}	0	0.25	0.2	0.4
	{R,Y,G}	0	0.3	0.2	0.4

ψ:{C,E}		ξ			
		{}	{N}	{O}	{N,O}
ϕ	{}	0	0	0	0
	{R}	0	0.07	0.06	0.1
	{Y}	0	0.1	0.3	0.4
	{G}	0	0.2	0.5	0.6
	{R,Y}	0	0.1	0.35	0.4
	{R,G}	0	0.2	0.5	0.6
	{Y,G}	0	0.25	0.65	0.9
	{R,Y,G}	0	0.3	0.7	1

essential properties. But it also does not introduce arbitrarily more "certainty" into the result of composition (in the sense of Maximum Uncertainty Principle).

5 Conclusions

As a preliminary result, we propose a definition of composition operator based on "minimum product". Its main advantage consists in the compliance with a basic postulate, the Maximum Uncertainty Principle. But still the following questions appear to be open:

- Duality - isn't the duality axiom too strong? How to handle it in case of multivariate distributions? (Too complicated constraints in case of multivariate distributions and search of complementary events.)
- Why is the product (in any variant) stated as an axiom?
- Maximum uncertainty principle restricts the "definition" of product to its minimum shape. Doesn't it follow completely already from the maximum uncertainty principle?
- Is the "definition" of the product rich enough to describe non-trivial dependence structures analogically to the situation in probabilistic models?
- Does the definition of composition show useful properties?

It is obvious that the attempt to define the operator of composition opened a wide field of unanswered questions concerning the philosophic fundamentals of Liu's Uncertainty Theory and thus provides enough space for further research.

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Generative Adversial Networks, a 2019 review

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Abstract

The generative adversial networks (GANs) represent an exciting concept that lies at the borderline between probabilistic modeling and machine learning. Despite the short history (since 2014) the GANs have caused a dramatic shift in several machine learning fields, however, mostly in generating photorealistic pictures of different objects. The basic idea the GANs are based on is conceptually simple, however, concrete implementations are still partly an art rather than established science and are tightly interconnected with programming in deep-learning frameworks such as TensorFlow or PyTorch to mention the most popular ones. The purpose of this contribution is to present a brief review of the GAN implementations milestones that paved the way to their current success. The list is not exhaustive as hundreds of implementations are available in several GANs ZOOs, see e.g., [5], but we try to be representative.

The seminal paper by Goodfellow at al. [4] introduced the concept. Despite rooted in an experimental work, the paper contains a rigorous proof for that the GAN minimax criterion leads in the end to sampling from the distribution the real data comes from. The presented examples in the paper are delivered for the MNIST, TFD and CIFAR-10 databases and the implementations work well with only basic neural network architectures such as the perceptrons. While the results are not so impressive they show that the GANs concept is viable and delivers, which has been fully proved in the subsequent works.

An important follow up work concerns introducing conditioning into the GANs area [9]. Extending noise input with a conditioning vector enabled generation of delimited classes of objects. The canonical example is generating individual digits from the MNIST dataset. A substantial move in the field came when GANs incorporated deep convolutional architectures [10]. The proposed architecture, called DCGAN, introduced upsampling operation that increases a resolution of the images generated from a low dimensional noise. DCGAN was used to generate pictures from the LSUN datasets [11].

The paper also introduces a vector arithmetic for generated images of human faces. Whilst faces are in some sense smooth objects, i.e., easier for generation via continuous transformations, DCGAN proves to be effective also in generation pictures based on the LSUN bedrooms datasets which frequently contains sharp edges (furniture or window edges) showing universality of the GAN concept.

Efforts have been made to theoretically guarantee stability of GAN learning. One of the main contribution in this direction was incorporating the Wasserstein distance in learning criteria leading to the Wasserstein GAN [1]. Building on the Wasserstein GAN ideas, the BEGAN architecture was proposed in [2] by controlling the equilibrium state between generator and discriminator during GAN training. The BEGAN architecture brought an improved quality of generated pictures namely for human faces resulting from learning the CelebA dataset.

A disruptive moment in the field was the introduction of progressive learning in the PGGAN [6] by the NVIDIA research team. The key idea in their approach is a gradual increasing of both generator and discriminator networks during learning. In effect, the photorealistic portraits were generated in 1024x1024 resolution. As a byproduct, the NVIDIA team created the HQ CelebA dataset of 1024x1024 photographs [7] that were used in PGGAN learning and represent a new standard dataset for general machine learning purposes.

A long-term goal in the GAN learning area, was "conquering" the ImageNet dataset. That is, to use the ImageNet dataset to construct a single GAN that would be able to conditionally generate samples from all 1000 categories the ImageNet consists from. The achievement of this goal was announced by the Google DeepMind team in contribution called BigGAN [3] which was presented at the ICLR 2019 conference.

The last milestone we mention is called StyleGAN [8] which comes again from the NVIDIA research. The paper introduces an alternative generator architecture drawing on style transfer methodology. The new architecture leads to an unsupervised separation of high-level attributes such as pose or identity when learning to generate human faces pictures and enables to conditionally generate images for different attributes.

In our contribution, we present examples generated by the above architectures so that the quality progress over time can be identified. We also mention the current challenges the field of GANs is facing and possible directions of future development in this flourishing area of machine learning.

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A Step towards Upper-bound of Conflict of Belief Functions based on Non-conflicting Parts

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Abstract

This study compares the size of conflict based on non-conflicting parts of belief functions *Conf* with the sum of all multiples of bbms of disjoint focal elements of belief functions in question. In general, we make an effort to reach a simple upper bound function for *Conf*. (Nevertheless, the maximal value of conflict is, of course, equal to 1 for fully conflicting belief functions). We apply both theoretical research using the recent results on belief functions and also experimental computational approach here.

Keywords: Belief functions, Dempster-Shafer theory, Uncertainty, Conflicting belief masses, Conflict between belief functions, Hidden conflict.

1 Introduction

Belief functions representing an uncertain and/or incomplete, imperfect information about the object of interest may be, of course, in mutual conflict. The classic definition of conflict between belief functions is equivalent to the sum of all multiples of conflicting belief masses of individual belief functions [17]; i.e. the belief mass assigned to the empty set when non-normalized conjunction combination rule is considered (frequently denoted by $m_{\textcircled{\cap}}(\emptyset)$). After this measure was observed to be inadequate for a correct representation of conflict between belief functions [1, 14], several different measures were introduced in last dozen years,

e.g. [6, 7, 12, 13, 14, 15, 16, 19]. Conflict between belief functions is usually assumed to be less or equal to the belief mass appearing on the empty set $m_{\textcircled{\cap}}(\emptyset)$.

One of the progressive current alternative conflict measures of the conflict between belief functions is based on their non-conflicting parts [7]. Despite the original assumption, positive conflict was observed there even in situations when the previously mentioned conflict measures were zero and belief functions in question were considered to be non-conflicting. These so-called hidden conflicts were analyzed and presented in [8, 11]. In this paper we try to give a simple upper-bound function of conflict based on non-conflicting parts and also of previous measures of conflict, to obtain an improved general assumption for conflict measures.

We apply here both theoretical approach using our recent results on degrees of hidden conflicts [11] and of degrees of non-conflictness [10] and also experimental computational approach continuing our computations from [8, 9].

2 Preliminaries

We assume classic definitions and basic notion from the theory of *belief functions* [17] on finite exhaustive frames of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. $\mathcal{P}(\Omega) = \{X | X \subseteq \Omega\}$ is a *power-set* of Ω.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0, 1]$ such that $\sum_{A \subseteq \Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed.

A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, such that $Bel(A) = \sum_{\emptyset \neq X \subseteq A} m(X)$. A *plausibility function* $Pl : \mathcal{P}(\Omega) \longrightarrow [0, 1]$, $Pl(A) = \sum_{\emptyset \neq A \cap X} m(X)$. Because there is a unique correspondence among m and corresponding Bel and Pl, we often speak about m as of a belief function.

A *focal element* is a subset of the frame of discernment $X \subseteq \Omega$, such that $m(X) > 0$; if $X \subsetneq \Omega$ then it is a *proper focal element*. If all focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function (BBF)*; in fact, it is a probability distribution on Ω. If there are only focal elements such that $|X| = 1$ or $|X| = n$ we speak about *quasi-Bayesian BF (qBBF)*. In the case of $m(\Omega) = 1$ we speak about *vacuous BF*. In the case of $m(X) = 1$ for $X \subset \Omega$ we speak about *categorical BF*. If $m(X) > 0$ for $X \subset \Omega$ and $m(\Omega) = 1 - m(X)$ we speak about *simple support BF*. If all focal elements have a non-empty intersection, we speak about *a consistent BF*; and if all of them are nested, about *a consonant BF*.

Dempster's (normalized conjunctive) rule of combination $\oplus$: $(m_1 \oplus m_2)(A) = \sum_{X \cap Y = A} K m_1(X)\, m_2(Y)$ for $A \neq \emptyset$, where $K = \frac{1}{1-\kappa}$, $\kappa = \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y)$, and $(m_1 \oplus m_2)(\emptyset) = 0$, see [17]. Putting $K = 1$ and $(m_1 \textcircled{\cap} m_2)(\emptyset) = \kappa = m_{\textcircled{\cap}}(\emptyset)$ we obtain the *non-normalized conjunctive rule of combination* $\textcircled{\cap}$, see e. g. [18].

Smets' *pignistic probability* is given by $BetP(\omega_i) = \sum_{\omega_i \in X \subseteq \Omega} \frac{1}{|X|} \frac{m(X)}{1 - m(\emptyset)}$, see e.g. [18]. *Normalized plausibility of singletons*[1] of Bel is a probability distribution

[1] Plausibility of singletons is called *contour function* by Shafer in [17], thus $Pl\_P(Bel)$ is a

Pl_P such that $Pl\_P(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega\in\Omega} Pl(\{\omega\})}$ [3, 4].

3 Conflicts of Belief Functions

Original Shafer's definition of the conflict measure between two belief functions [17] is the following: $\kappa = \sum_{X\cap Y=\emptyset} m_1(X)m_2(Y) = (m'\textcircled{\cap}m'')(\emptyset) = m_{\textcircled{\cap}}(\emptyset)$, more precisely its transformation $log(1/(1-\kappa))$.

After appearing that $m_{\textcircled{\cap}}(\emptyset)$ does not correctly represent conflict between BFs [1, 14] a series of alternative approaches and measures of conflicts have appeared in last dozen years, e. g. [2, 6, 7, 12, 14, 15]. Alternative approaches are often somehow related to $m_{\textcircled{\cap}}(\emptyset)$ or use it as one of its components [14].

In 2010, Daniel distinguished internal conflict inside an individual BF from the conflict between them [5] and pointed out that $m_{\textcircled{\cap}}(\emptyset)$ contains both individual internal conflicts of BFs and conflict between them. Thus the usual assumption or property of measures of conflict to be less or equal to $m_{\textcircled{\cap}}(\emptyset)$ seemed to be natural.

Finally, Daniel's *conflict based on non-conflicting parts of BFs* was introduced in [7]. This last-mentioned measure motivated our research of hidden conflict [9], hidden auto-conflict [8] and also current research of degrees of non-conflictness [10].

A conflict of BFs Bel', Bel'' based on their non-conflicting parts Bel'_0, Bel''_0 is defined by the expression $Conf(Bel', Bel'') = (m'_0\textcircled{\cap}m''_0)(\emptyset)$, where non-conflicting part Bel_0 (of a BF Bel) is unique consonant BF such that $Pl\_P_0 = Pl\_P$ (normalized plausibility of singletons corresponding to Bel_0 is the same as that corresponding to Bel); m_0 is a bba related to Bel_0. For an algorithm to compute Bel_0 see [7].

This measure of conflict between BFs in correspondence to Daniel's approach from [5] does not include internal conflict of individual BFs. And Theorem 4 from [7] claims that

$$Conf(Bel^i, Bel^{ii}) \leq (m^i\textcircled{\cap}m^{ii})(\emptyset) \qquad (*)$$

holds true for arbitrary BFs Bel^i, Bel^{ii} given by bbas m^i, m^{ii} on any finite frame of discernment Ω_n. Nevertheless, during later analysis of $Conf$ properties counter-examples against general validity of $(*)$ have appeared, for some of them see the next Section,

Similarly to plausibility conflict, measure $Conf$ respects plausibilities equivalent to the BFs; and it better generalises the original idea to general frame Ω_n.

4 Counter-Examples against General Validity of Inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$

There are plenty of counter-examples against general validity of inequality $(*)$, thus against $Conf \leq m_{\textcircled{\cap}}(\emptyset)$. Counter-examples have started to appear when the

normalization of contour function in fact.

first hidden conflicts had been observed. Any hidden conflict is a counterexample against it. We can start with the first and simple Introductory Example from [8, 9] on Ω_3 and Little Angel example from [9], for both the examples see also [11]).

Example 1. **Introductory example.** Let us assume two simple consistent belief functions Bel' and Bel'' on $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ given by the bbas $m'(\{\omega_1, \omega_2\}) = 0.6$, $m'(\{\omega_1, \omega_3\}) = 0.4$, and $m''(\{\omega_2, \omega_3\}) = 1.0$.

We can display focal elements of BFs Bel' and Bel'' on Figure 1.

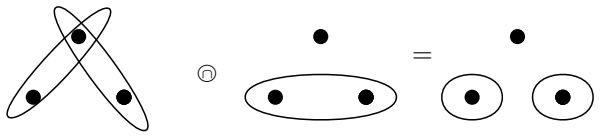

Figure 1: Introductory Example: focal elements of m', m'', and of $m' \circledcirc m''$.

ω_1 is in both the focal elements of Bel', thus $Pl'(\{\omega_1\}) = 0.6 + 0.4 = 1$, and the other two singletons each in the only focal element, thus simply $Pl'(\{\omega_2\}) = 0.6$, $Pl'(\{\omega_3\}) = 0.4$ and after the normalization $Pl\_P' = (0.5, 0.3, 0.2)$. For Bel'' analogously $Pl''(\{\omega_2\}) = Pl''(\{\omega_3\}) = 1.0$ and $Pl\_P'' = (0.0, 0.5, 0.5)$. Thus non-conflicting parts of the BFs are given by the following bbms: $m'_0(\{\omega_1\}) = \frac{0.5-0.3}{0.5} = \frac{2}{5} = 0.4$, $m'_0(\{\omega_1, \omega_2\}) = \frac{0.3-0.2}{0.5} = \frac{1}{5} = 0.2$, and $m'_0(\{\omega_1, \omega_2, \omega_3\}) = \frac{0.2}{0.5} = 0.4$, and $m''_0(\{\omega_2, \omega_3\}) = m''(\{\omega_2, \omega_3\}) = 1.0$.
Hence we obtain $Conf(Bel', Bel'') = m'_0(\{\omega_1\})m''_0(\{\omega_2, \omega_3\}) = 0.4 \cdot 1.0 = 0.4 > 0 = (m' \circledcirc m'')(\emptyset)$.

Example 2. **Little Angel example** Let assume belief functions Bel^i and Bel^{ii} on $\Omega_5 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ given by the bbas $m^i(\{\omega_1, \omega_2, \omega_3\}) = 0.1$, $m^i(\{\omega_1, \omega_2, \omega_3, \omega_4\}) = 0.3$, $m^i(\{\omega_1, \omega_3, \omega_4, \omega_5\}) = 0.6$, and $m^{ii}(\{\omega_2, \omega_3.\omega_4, \omega_5\}) = 1.0$.

Analogously to the previous case, we can display focal elements of BFs Bel^i and Bel^{ii} on Figure 2.

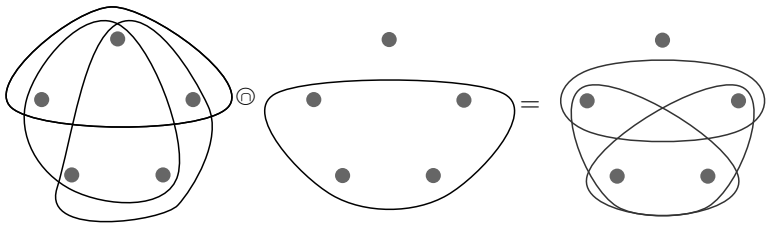

Figure 2: Litte Angel Example: focal elements of m^i, m^{ii}, and of $m^i \circledcirc m^{ii}$.

Analogously to the previous case we obtain the following plausibility of singletons: $Pl^i(1.0, 0.4, 0.9, 0.9, 0.7)$, $Pl\_P^i(\frac{10}{39}, \frac{4}{39}, \frac{9}{39}, \frac{9}{39}, \frac{7}{39})$ and $m^i_0(\{\omega_1\}) = \frac{10-9}{10} = 0.1$, $m^i_0(\{\omega_1, \omega_3, \omega_4\}) = \frac{9-7}{10} = 0.2$, $m^i_0(\{\omega_1, \omega_3, \omega_4, \omega_5\}) = \frac{7-4}{10} = 0.3$, $m^i_0(\{\omega_1, \omega_2,$

$\omega_3, \omega_4, \omega_5\}) = \frac{4}{10} = 0.4$. For Bel^{ii} there is $m_0^{ii}(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = m^{ii}(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = 1.0$. Hence we obtain $Conf(Bel^i, Bel^{ii}) = m_0^i(\{\omega_1\})m_0^{ii}(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = 0.1 \cdot 1 = 0.1$.

Both these examples are simple with a few focal elements only. Nevertheless, we can find plenty of the other examples moving small belief masses from the original focal elements to the other subset of the frame and create new ones:

Example 3. **Modified Introductory example.** Let us suppose belied functions Bel', Bel'' given by the modified bbas m' and m'', moving parts of the original bbms to singletons and to entire Ω_3 as it follows:

$m'(\{\omega_1\})$	$= 0.1$	$m''(\{\omega_1\})$	$= 0.1$
$m'(\{\omega_2\})$	$= 0.1$	$m''(\{\omega_2\})$	$= 0.1$
$m'(\{\omega_3\})$	$= 0.1$	$m''(\{\omega_3\})$	$= 0.1$
$m'(\{\omega_1, \omega_2\})$	$= 0.4$	-	
$m'(\{\omega_1, \omega_3\})$	$= 0.2$	-	
-		$m''(\{\omega_2, \omega_3\})$	$= 0.6$
$m'(\{\omega_1, \omega_2, \omega_3\})$	$= 0.1$	$m''(\{\omega_1, \omega_2, \omega_3\})$	$= 0.1$

After this modification we obtain $Pl\_P' = (\frac{8}{18}, \frac{6}{18}, \frac{4}{18})$, $Pl\_P'' = (\frac{2}{18}, \frac{8}{18}, \frac{8}{18})$, and further $m_0'(\{\omega_1\}) = \frac{8-6}{8} = \frac{2}{8} = 0.25$, $m_0'(\{\omega_1, \omega_2\}) = \frac{6-4}{8} = \frac{2}{8} = 0.25$, and $m_0'(\{\omega_1, \omega_2, \omega_3\}) = \frac{4}{8} = 0.5$, and $m_0''(\{\omega_2, \omega_3\}) = \frac{8-2}{8} = 0.75$, $m_0''(\{\omega_1, \omega_2, \omega_3\}) = \frac{2}{8} = 0.25$.
Hence we obtain $Conf(Bel', Bel'') = m_0'(\{\omega_1\})m_0''(\{\omega_2, \omega_3\}) = 0.25 \cdot 0.75 = 0.1875$.
$(m' \textcircled{\scriptsize}\, m'')(\emptyset) = 6 \times 0.1 \cdot 0.1 + 0.1 \cdot 0.6 + 0.4 \cdot 0.1 + 0.2 \cdot 0.1 = 0.06 + 0.06 + 0.04 + 0.02 = 0.18$.
Hence $Conf(Bel', Bel'') = 0.1875 > 0.1800 = (m' \textcircled{\scriptsize}\, m'')(\emptyset)$.

Example 4. **Modified Little Angel example** For following modification of Little Angel BFs we obtain counter-example again:

$m^i(\{\omega_1\})$	$= 0.05$	-	
$m^i(\{\omega_2\})$	$= 0.05$	$m^{ii}(\{\omega_2\})$	$= 0.05$
$m^i(\{\omega_1, \omega_2\})$	$= 0.05$	-	
$m^i(\{\omega_2, \omega_4\})$	$= 0.05$	-	
-		$m^{ii}(\{\omega_3, \omega_4\})$	$= 0.10$
$m^i(\{\omega_1, \omega_2, \omega_5\})$	$= 0.10$	-	
$m^i(\{\omega_2, \omega_3, \omega_4\})$	$= 0.05$	-	
$m^i(\{\omega_1, \omega_2, \omega_3, \omega_4\})$	$= 0.20$	-	
$m^i(\{\omega_1, \omega_3, \omega_4, \omega_5\})$	$= 0.40$	-	
-		$m^{ii}(\{\omega_2, \omega_3, \omega_4, \omega_5\})$	$= 0.80$
$m^i(\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\})$	$= 0.05$	$m^{ii}(\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\})$	$= 0.05$

where $Conf(Bel', Bel'') = 0.1114551$ while $(m' \textcircled{\scriptsize}\, m'')(\emptyset) = 0.0875$.

After observation of the original examples, we had a working hypothesis of validity of equality $(*)$ for all quasi Bayesian BFs on a general finite frame of discernment, unfortunately, instead of proving the hypothesis we have found several counterexamples for both qBBF and even for Bayesian BFs already on four-element

frame of discernment Ω_4. We have used a method described in Section 8. The first counterexample found on Ω_4 is shown in the next more general Example with $\varepsilon = 0$ in which case both BFs are Bayesian and $Conf(Bel^i, Bel^{ii}) = 0.984375$ while $(m^i \circledcirc m^{ii})(\emptyset) = 0.98$.

Example 5. **8-1-1-0** Assume the following class of BFs on Ω_4:

$$\begin{array}{rlrl} m^i(\{\omega_1\}) & = 0.1 & m^{ii}(\{\omega_1\}) & = 0.1 \\ & & m^{ii}(\{\omega_2\}) & = 0.8 - \varepsilon \\ m^i(\{\omega_3\}) & = 0.8 - \varepsilon & & \\ m^i(\{\omega_4\}) & = 0.1 & m^{ii}(\{\omega_4\}) & = 0.1 \\ m^i(\{\omega_1, \omega_2, \omega_3, \omega_4\}) & = \varepsilon & m^{ii}(\{\omega_1, \omega_2, \omega_3, \omega_4\}) & = \varepsilon \end{array}$$

Because the inequality $(*)$ holds for the majority of qBBFs on Ω_4, i.e. for $n = 4$, size of belief mass moved to the entire frame must be rather small. Indeed, note that in case of $\varepsilon < 0.008$ the inequality $(*)$ does not hold (the exact bound is slightly higher). For $\varepsilon = 0.008$ we obtain $Conf(Bel^i, Bel^{ii}) = 0.964475 > 0.964064 = (m^i \circledcirc m^{ii})(\emptyset)$.

Any of the above-presented examples can be easily extended for a greater frame of discernment. For an extension of Example 8-1-1-0 to 10-element frame of discernment Ω_{10} see m^{i-0}, m^{ii-0} in Example 8-small-small: Example 6, Section 6.

5 Validity of $Conf \leq m_{\textcircled{\cap}}(\emptyset)$

We can start from the simplest case of 2-element frame of discernment, which had motivated too strong version of the statement about $(*)$ in Belief'14 [7]:

Lemma 1 *Inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ holds true for arbitrary BFs on a 2-element frame of discernment Ω_2.*

Proof. Let us denote $(a_i, b_i) = (m_i(\{\omega_1\}), m_i(\{\omega_2\})$, thus $m_i(\{\omega_1, \omega_2\}) = 1 - a_i - b_i$. $m(\emptyset) = (m_1 \circledcirc m_2)(\emptyset) = a_1 b_2 + a_2 b_1$. Let us suppose $a_1 \geq b_1$. If also $a_2 \geq b_2$ then both maximal plausibilities are higher for ω_1, thus $Conf(m_1, m_2) = 0 \leq m(\emptyset)$. Hence there still remain to prove the case $a_2 \leq b_2$. There it holds: $Pl_i = (1 - b_i, 1 - a_i)$ and $Pl\_P_i = (\frac{1-b_i}{2-a_i-b_i}, \frac{1-a_i}{2-a_i-b_i})$. $m_{01}(\{\omega_1\}) = a_{01} = \frac{a_1 - b_1}{1-b_1}$, $m_{01} = (a_{01}, b_{01}) = (\frac{a_1-b_1}{1-b_1}, 0)$, and analogously $m_{02} = (a_{02}, b_{02}) = (0, \frac{b_2-a_2}{1-a_2}.)$. Thus $Conf(m_1, m_2) = (\frac{a_1-b_1}{1-b_1})(\frac{b_2-a_2}{1-a_2})$. Hence it remains to verify $(\frac{a_1-b_1}{1-b_1})(\frac{b_2-a_2}{1-a_2}) \leq a_1 b_2 + a_2 b_1$. We can show that $(\frac{a_1-b_1}{1-b_1}) \leq a_1$: $(a_1 - b_1) \geq a_1 1 - b_1) = a_1 - a_1 b_1$, $-b_1 \leq -a_1 b_1$, $0 \leq b_1(1 - a_1)$, what follows $0 \leq a, b \leq 1$, analogously we can show that $(\frac{b_2-a_2}{1-a_2}) \leq b_2$. Hence $(\frac{a_1-b_1}{1-b_1})(\frac{b_2-a_2}{1-a_2}) \leq a_1 b_2 \leq a_1 b_2 + a_2 b_1$.

Lemma 2 *Inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ holds for any pair of consonant BFs on any finite Ω_n.*

Proof. The statement follows the fact that there is the unique consonant non-conflicting part of a BF. Consonant BFs are because of this uniqueness equal to their non-conflicting parts. Hence $Conf(Bel^i, Bel^{ii}) = (m_0^i \circledcirc m_0^{ii})(\emptyset) = (m^i \circledcirc m^{ii})(\emptyset)$, thus $Conf = m_{\circledcirc}(\emptyset)$, hence inequality holds true.

Corollary 3 *(i) Inequality $Conf \leq m_{\circledcirc}(\emptyset)$ holds for any pair of categorical BFs on any finite frame of discernment Ω_n.*

(ii) Inequality $Conf \leq m_{\circledcirc}(\emptyset)$ holds for any pair of simple support BFs on any finite frame of discernment Ω_n.

Proof. Both categorical and simple support belief functions are consonant, thus both (i) and (ii) are special cases of Lemma 2.

Unfortunately, as we have seen in Example 8-1-1-0, inequality $(*)$ does not holds either for two arbitrary quasi Bayesian BFs on Ω_4. Nevertheless, We have no counter-example against validity of $(*)$ for qBBFs on Ω_3 but we also do not have a complete proof of its validity yet. That is why we moved the issue of qBBFs on Ω_3 to the next section about hypotheses. To complete this Section, we have to mention the following trivial observation:

Observation 4 *If one of the belief functions in question is vacuous, inequality $Conf \leq m_{\circledcirc}(\emptyset)$ always holds. ($Conf = 0 = m_{\circledcirc}(\emptyset)$ in that cases.)*

6 Hypotheses

Hypothesis 5 *Inequality $Conf \leq m_{\circledcirc}(\emptyset)$ holds true for any couple of quasi Bayesian belief functions on any 3-element frame Ω_3.*

Partial proof. If we want to find a proof analogous to that for Ω_2, there is no problem with $(m^i \circledcirc m^{ii})(\emptyset)$: formula for its computation from input bbms is always the same for given cardinality n of Ω_n. This is different for computation of value $Conf(Bel^i, Bel^{ii})$, where different focal elements appear in corresponding Bel_0^i, Bel_0^{ii}, also focal element of cardinality 2, thus there is not only higher complexity of formula for higher n, but the number of different formulas for different orders of values of plausibility of singletons (for qBBFs equal to order of input bbms). Moreover, neither analogy of proof on Ω_2 has not been found for any of the cases of formulas on Ω_3. Nevertheless, we have some kind of proof for some special cases of input qBBFs.

Two simplified cases have been already proven with the usage of *WolframAplha tool: https://www.wolframalpha.com* just by checking if the formula corresponding to the inequality $(*)$ has a solution in the $[0, 1]$ interval of respective variables. Nevertheless, a complete list of formulae for the general case has not been formulated yet, thus either proved.

Let us start from the case of Bayesian BFs with only two positive values: $(a, 1-a, 0|0)$, $(0, 1-a, a|0)$ where the notation corresponds to $(m(\omega_1), m(\omega_2), m(\omega_3)|m(\Omega_3))$, with maximal values assigned to different singletons (otherwise $Conf = 0$). If $a \geq 1-a$, i.e. $a \geq 1/2$, then we obtain: $Conf : \left(\frac{2a-1}{a}\right)^2 + 2\left(\frac{2a-1}{a}\right)\left(\frac{1-a}{a}\right)$ and $m(\emptyset) = a^2 + 2a(1-a) = 2a - a^2$.

Nevertheless, the inequality corresponding to the opposite of $(*)$: $\left(\frac{2a-1}{a}\right)^2 + 2\left(\frac{2a-1}{a}\right)\left(\frac{1-a}{a}\right) - 2a + a^2 > 0$ has a solution only for $a \notin [-1, 1]$ and therefore no counter-example can exists in this case.

Analogously we can consider BBFs one with three different values and the other with opposite order of the same bbms, thus $(a, b, 1-a-b|0)$, $(1-a-b, b, a|0)$. In case of $a > b > 1-a-b$ we obtain: $\left(\frac{a-b}{a}\right)^2 + 2\left(\frac{a+2b-1}{a}\right)\left(\frac{a-b}{a}\right) - 2a^2 - 2ab + b^2 + 2a - 1 > 0$ that is necessary to hold for a counter-example, but again, it has not solution for $0 <= b <= a <= 1-b$. Therefore NO counter-example can exist here.

In the completely general case for qBBFs on Ω_3 it is necessary to verify several inequalities with 6 variables. As an example we can present the inequality $(\frac{a-b}{1-b-c})(\frac{f-e}{1-d-e}) + (\frac{a-b}{1-b-c})(\frac{e-d}{1-d-e}) + (\frac{b-c}{1-b-c})(\frac{1-d-2e}{1-d-e}) - (ae+af+bd+bf+cd+ce) \leq 0$, which should be verified for the case $(a, b, c|1-a-b-c)$ and $(c, d, e|1-c-d-e)$ where $a > b > c$ and $c < d < e$.

In the future, we would like to analogously check all the possible Ω_3 cases.

Hypothesis 6 *Inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ holds true for any couple of quasi Bayesian belief functions, having all singletons ($m(\{\omega_i\}) > 0$ for any $\omega_i \in \Omega_n$) on any finite frame of discernment Ω_n.*

Arguments for this hypothesis are as follows:

(i) We have not found any counter-example on Ω_n for $n <= 5$ yet.

(ii) When moving some positive mass to any singletons in Example 8-small-small, $Conf$ decreased bellow $m(\emptyset)$, thus counter-example disappears. This was checked both for BBF and qBBF counter-examples with BFs without some singletons on Ω_{10}.

(iii) It is a sort of generalization of the previous hypothesis.

Example 6. **Example 8-small-small** Let suppose BBFs m^{i-0} and m^{ii-0} which are extensions of BFs from Example 8-1-1-0 to Ω_{10}. And their further extension to qBBFs (thus qBBFs 'with 0', i.e. without some singleton focal elements, m^{i-2} and m^{ii-2}; (zeros are not typed to be more visible). We can verify, that these are really counter-examples against general validity of $Conf \leq m_{\textcircled{\cap}}(\emptyset)$:
$Conf(m^{i-0}, m^{ii-0}) = 0.9982937 > 0.995608 = (m^{i-0} \textcircled{\cap} m^{ii-0})(\emptyset)$ and also
$Conf(m^{i-2}, m^{ii-2}) = 0.9884347 > 0.98788 = (m^{i-2} \textcircled{\cap} m^{ii-2})(\emptyset)$.
When we remove some belief masses to missing singletons we obtain BBFs 'without zero' m^{i-1}, m^{ii-1} and qBBFs 'without zero' m^{i-3}, m^{ii-3} in both these cases counter-examples against $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ disappear:

$Conf(m^{i-1}, m^{ii-1}) = 0.9841313 < 0.986424 = (m^{i-1} \textcircled{\cap} m^{ii-1})(\emptyset)$and also
$Conf(m^{i-3}, m^{ii-3}) = 0.9744128 > 0.978536 = (m^{i-3} \textcircled{\cap} m^{ii-3})(\emptyset)$.

$X \subseteq \Omega_{10}$	BBF with 0		BBF without 0		qBBF with 0		qBBF without 0	
$X \subseteq \Omega_{10}$	m^{i-0}	m^{ii-0}	m^{i-1}	m^{ii-1}	m^{i-2}	m^{ii-2}	m^{i-3}	m^{ii-3}
$\{\omega_1\}$	0.800		0.800	0.006	0.800		0.800	0.006
$\{\omega_2\}$	0.040	0.012	0.040	0.010	0.040	0.012	0.038	0.010
$\{\omega_3\}$	0.034	0.016	0.034	0.014	0.034	0.014	0.032	0.014
$\{\omega_4\}$	0.030	0.020	0.030	0.018	0.030	0.018	0.030	0.018
$\{\omega_5\}$	0.026	0.022	0.026	0.022	0.026	0.022	0.026	0.022
$\{\omega_6\}$	0.022	0.026	0.022	0.026	0.022	0.026	0.022	0.026
$\{\omega_7\}$	0.020	0.030	0.018	0.030	0.018	0.030	0.018	0.030
$\{\omega_8\}$	0.016	0.034	0.014	0.034	0.014	0.034	0.014	0.032
$\{\omega_9\}$	0.012	0.040	0.010	0.040	0.012	0.040	0.010	0.028
$\{\omega_{10}\}$		0.800	0.006	0.800		0.800	0.006	0.800
Ω_{10}					0.004	0.004	0.004	0.004

Thus we have a couple of similar BFs, one without some singleton, which is counter-example and the other with all singletons (positive bbms for all elements of the frame of discernment), which is not a counterexample. The same we have both for general qBBFs and BBFs. Note that there are also extensions of m^{i-0}, m^{ii-0} to qBBFs which are not counter-examples either 'with 0' or 'without 0', nevertheless these are not interesting for us w.r.t. to Hypothesis 6.

7 Open Problems

There are plenty of open problems related to this topic, especially (i) to decide whether the hypotheses from the previous section hold true or not, and (ii) for which sets of belief functions inequality $(*)$ holds true and for which does not.

In the context of Lemma 2 it seems to be interesting to decide, whether the inequality holds also for any couple of consonant BFs. Nevertheless, this is not the case, as in both Examples 1 and 2 both BFs are consonant and therefore we already have counter-examples on Ω_3 and Ω_5.

A completely different question we did not study so far is whether inequality $(*)$ holds for separable support belief functions (i.e. Dempster's $\oplus$-sum of simple support functions). Note that it has a relation to statement (ii) in Corollary 3.

Because the inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ holds only for some types of BFs, we have to weaken the inequality for more general validity. Perspective/prospective seems to be question of validity $Conf \leq (\textcircled{\cap}_1^k(m_1 \textcircled{\cap} m_2))(\emptyset)$ for a convenient k. $(\textcircled{\cap}_1^k(m_1 \textcircled{\cap} m_2))(\emptyset)$ is related to the hidden conflicts and looking for full non-conflictness [8, 9]. We can see that $(*)$ is this inequality for $k = 1$. Due to a new parameter k in the inequality this is rather complex challenging topic for our future research.

8 Computation

We have performed many computational experiments to find a counterexample to $(*)$ for different classes of BFs on different Ω_n. To do so, we have used R-Project, a free tool for statistical computing and we have implemented all procedures needed to calculate various conflicts of BFs. BFs are represented using an object based on several database tables and we have employed the effective implementation of the join operation for relational databases.

The plan was to systematically search the whole space of BFs of a certain class. In our search for counterexamples we took BFs being on a grid defined using a fixed step for bbm values taken in consideration (e.g. 0.1). Then we generated all BFs with bbms which are multiples of the step. As an example of such BFs you can take m' and m'' from Example 1.

The idea of the grid is quite simple. In the case of $n = 3$ and general BFs you have 8 possible focal elements, i.e. up to 8 bbms. Assume step 0.1. Then we can divide the total belief mass of 1 into 10 pieces. Find all sets of integers that sum to 10 and limit your search to sets of cardinality 8 and less. Then, to find all BFs with bbms divisible by the step, you just take all permutations of respective sets of integers multiplied by the step and use them as bbms.

Because of the exponential increase in the number of focal elements and the number of permutations, the amount of different BFs is overwhelming even for small n regardless of the step. We were only able to go through a few classes. In the case of the other classes, we will have to run through the grid at random. Another method, which we successfully applied in Examples 5 and 6, is to identify an area with a chance of success and pass it with another step.

To illustrate the calculations, let us provide several numbers. Note that in case of $n = 3$ and step 0.1, there are 8.046 general BFs and 552 qBBFs. By decreasing the step to 0.01 we have 10^{11} BFs and 8.037 qBBFs. Note that out of these $64 \cdot 10^6$ combinations of qBBFs ($n = 3$, step: 0.01) 706.751 represent counterexamples.

In case of $n = 4$, there are $5 \cdot 10^{13}$ general BFs with step 0.1 and 3.600 qBBFs. In case of qBBFs we can decrease the step to 0.025 (136.824 qBBFs), 0.02 (318.269 qBBFs), or 0.01 (4.598.126 qBBFs).

In case of $n = 5$ and step 0.1 there are 10^{37} general BFs and 4.200 qBBFs.

9 Conclusion

Motivated by appearing of counter-examples against general validity of inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$, relations of value of conflict between belief functions based on non-conflicting parts $Conf$ and of sum of all multiples of disjoint focal elements of belief functions in question $m_{\textcircled{\cap}}(\emptyset)$ have been analysed.

It has been proven that inequality $Conf \leq m_{\textcircled{\cap}}(\emptyset)$ holds for any couple of BFs on Ω_2, it was partially proved that it holds for any quasi Bayesian BFs on Ω_3 and hypothesis that the inequality holds for any couple of quasi Bayesian BFs with

positive values for all singletons on any finite frame Ω_n was formulated. Further, it was proven that it holds for any couple of consonant BFs on any finite frame Ω_n.

Besides it was shown, where the inequality does not holds: e.g. general BFs, on Ω_3, quasi Bayesian BFs without some singleton on Ω_4, etc. Several still open issues were formulated.

This study enables a better understanding of the measure of conflict $Conf$ and to understanding of conflicts between belief function in general.

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On the crossing number of join of graph of order six with path

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Abstract

The crossing number, $cr(G)$, of a simple graph G is the minimum number of edge crossings in a good drawing of G in the plane. In general, compute the crossing number for a given graph is a very difficult problem. The crossing numbers of a few families of graphs are known. One of them are join products of special graphs. Exact values of crossing numbers of the join products of graph G with discrete graph D_n, path P_n or cycle C_n are known for several graphs G. In the paper, we extend known results concerning crossing numbers for join of 6–vertex graph with path P_n.

1 Introduction

Let the *graph* G is a simple, undirected and connected with vertex set V and edge set E. A mapping that assings a point in the plane for each vertex and for each edge a continuous curve between its two endpoints is called a *drawing* D of the graph $G = (V, E)$. A *crossing* of two edges is the intersection of the interiors of the corresponding curves. The *crossing number*, $\mathrm{cr}(G)$, of a graph G is the minimum number of pairwise intersections of edges in any drawing of G in the plane. The drawing with a minimum number of crossings must be a *good drawing*, that means, each two edges have at most one point in common, which is either a commom end-vertex or a crossing.

Garey and Johnson proved [1] that computing the crossing number of a graph is an NP-complete problem. The exact values of crossings numbers are known for several special classes of graphs. One of them is a join products of two graphs. The join product $G_1 + G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is obtained from the vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. For $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of the graphs G_1, G_2, and the complete bipartite graph $K_{m,n}$. Let D_n consist on n isolated vertices, let P_n be the paths with n vertices and C_n be the cycle with n vertices. In the proofs of the paper, we will often use

the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

The exact values for crossing numbers of $G+P_n$ and $G+C_n$ for all graphs G of order at most four are given in [7], and the crossing numbers of the graphs $G+D_n$, $G+P_n$, and $G+C_n$ are also known for some graphs G of order five and six, see [16], [17], [5], [4], [8], [6], [9], [10], [11], [13], [12], and [14].

In this paper we extend these results by giving the exact values of the crossing numbers for join products for a special two graphs on six vertices with paths P_n.

Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$.
In the paper, is used the Kleitman's result published in [3] on crossing numbers of complete bipartite graphs. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6. \tag{1}$$

2 The crossing number of $G + P_n$

Let the graph G and its vertex notation is a graph in Figure 1(a). So, the graph G consists of one 6–cycle, $C_6(G)$, and one additional edge which together with the edges of $C_6(G)$ creates two 4–cycles. We consider the join product of G with the discrete graph D_n. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \ldots, t_n$, where avery vertex t_i, $i = 1, 2, \ldots, n$, is adjacent to every vertex of G. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the six edges incident with the vertex t_i. Thus, $T^1 \cup T^2 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{6,n}$ and

$$G + D_n = G \cup K_{6,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2}$$

The graph $G + P_n$ contains the graph $G + D_n$ and $n - 1$ edges $\{t_i, t_{i+1}\}$ for $i = 1, 2, \ldots, n-1$.

Let D be a good drawing of the graph $G+P_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i, see [2]. We use the notation (123456) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, t_iv_5, and t_iv_6. We emphasize that a rotation is a cyclic permutation. For $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, every subgraph $T^i \cup T^j$ of the graph $G+P_n$ is isomorphic with the graph $K_{6,2}$. We will study the minimum number of crossings between the edges of T^i and the edges of T^j in a subgraph $T^i \cup T^j$ induced in D of $G + P_n$ depending on the rotations $\mathrm{rot}_D(t_i)$ and $\mathrm{rot}_D(t_j)$.

D. R. Woodall [18] published that in the subdrawing of $T^i \cup T^j$ induced by D is $\mathrm{cr}(T^i, T^j) \geq 6$ if $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$. And, if $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ denotes the

minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$, then hold $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \leq \mathrm{cr}_D(T^i, T^j)$.

We will separate the subgraphs T^i for $i, j \in \{1, 2, \ldots, n\}$ into three subsets. First, $R_D = \{T^i : \mathrm{cr}_D(G, T^i) = 0\}$. Next, $S_D = \{T^i : \mathrm{cr}_D(G, T^i) = 1\}$. And the last subset contains every T^i which crosses G at least twice in D.

Let F^i denotes the subgraph $G \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \ldots, n\}$. Thus, any F^i is represented by $\mathrm{rot}_D(t_i)$. All cyclic permutations of six elements can be generated using the algorithm published in [15].

We are interested only in such drawings of G in which there is possibility to have $T^i \in R_D$. There is only one drawing of G without crossing shown in Figure 1(a). Assume a drawing D of the graph $G + P_n$ in which the edges of G does not cross each other. We can choose the vertex notation of the graph G in such a way as shown in Figure 1(a). So, in D there is the only possible configuration of F^i (see Figure 2(a)). There is only one drawing of the graph G with one crossing among its edges and with a possibility of an existence of a subgraph T^i which do not cross the edges of G. Assume now a good drawing D of the graph $G + P_n$ in which the edges of G cross once as shown in Figure 1(b). And, the last possibility is a drawing of the graph G with at least two crossings on edges of G. If there is $T^i \in R_D$, we have several such a drawings of G.

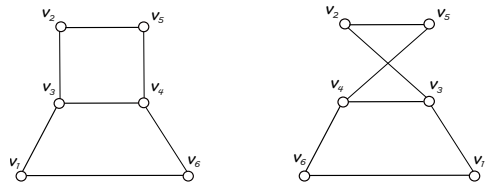

Figure 1: The graph G with the vertex notations

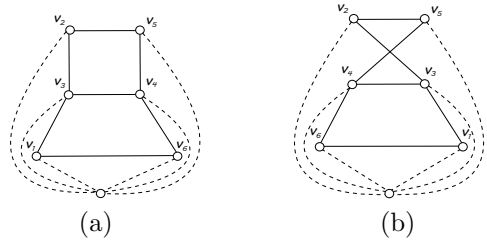

Figure 2: Drawing of the only possible configuration of the subgraph F^i

Theorem 1 $\mathrm{cr}(G + P_n) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor + 1$ *for* $n \geq 2$.

Proof:

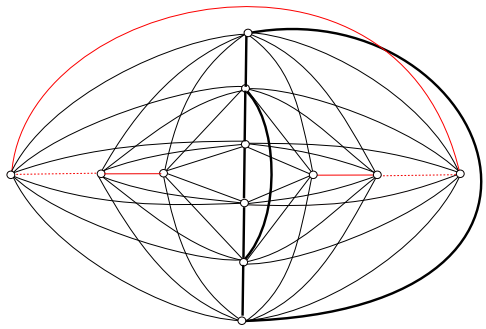

Figure 3: Drawing of $G + P_n$

There is a drawing of $G+P_n$ (see Figure 3) with $6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + 1$ crossings. Thus, we have $\mathrm{cr}(G + P_n) \leq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + 1$. We prove the reverse inequality by induction on n. Using algorithm on the website `http://crossings.uos.de/`, we can prove that the result is true for $n = 2$. Suppose now that, for $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G + P_n) < 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + 1, \tag{3}$$

and let

$$\mathrm{cr}_D(G + P_m) \geq 6\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor + 1 \qquad \text{for any integer } m < n. \tag{4}$$

As the graph $G + D_n$ is a subgraph of the graph $G + P_n$ and $\mathrm{cr}_D(G + D_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ (see [10]), then we assume that $\mathrm{cr}_D(G + P_n) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$. Thus, no edge of the path $P_n = (t_1, t_2, \ldots, t_n)$ is crossed in D.

First, we prove that the considered drawing D must be antipodal-free, that is $\mathrm{cr}_D(T^i, T^j) \neq 0$ for all i, j. As a contradiction suppose that, without loss of generality, $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. Since the graph $G \cup T^{n-1} \cup T^n$ contains $K_{4,3}$ as a subgraph, and $\mathrm{cr}(K_{4,3}) = 2$, we have $2 \leq \mathrm{cr}_D(G, T^{n-1} \cup T^n)$.

The fact that $\mathrm{cr}(K_{6,3}) = 6$ implies that any T^k, $k = 1, 2, \ldots, n-2$, crosses $T^{n-1} \cup T^n$ at least six times. So, for the number of crossings, in D, we have

$$\mathrm{cr}_D(G + P_n) = \mathrm{cr}_D\left(G + P_{n-2}\right) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{6,n-2}, T^{n-1} \cup T^n) +$$

$$+ \mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2\left\lfloor \frac{n-2}{2} \right\rfloor + 1 + 6(n-2) + 2 =$$

$$= 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + 1.$$

It contradicts that D is not antipodal-free.

Moreover, our assumption on D together with $\mathrm{cr}(K_{6,n}) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ implies that

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{6,n}) \leq 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Let us denote $r = |R_D|$ and $s = |S_D|$. Then,

$$\mathrm{cr}_D(G) + 0r + 1s + 2(n - r - s) \leq 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, $2r + s \geq 2n - 2\left\lfloor \frac{n}{2} \right\rfloor$. We will fixed one subgraph T^i.

Case 1: $\mathrm{cr}_D(G) = 0$.

We can choose the vertex notation of the graph as shown in Figure 1(a). At first, we prove, that $n \neq s$. If $n = s$, it means that for every $T^i \in S_D$ we have two possibilities.

(i) For every $i, j, i \neq j : \mathrm{cr}_D(T^i, T^j) \geq 3$.

Without lost of generality let $T^n \in S_D$ and let us fix $G \cup T^n$. We have

$$\mathrm{cr}(G + P_n) \geq \mathrm{cr}_D(K_{6,n-1}) + \mathrm{cr}_D(K_{6,n-1}, G \cup T^n) + \mathrm{cr}_D(G \cup T^n) \geq$$

$$\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-1) + 1 > 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

(ii) There is $T^i, T^j \in S_D$, $i \neq j$: $\mathrm{cr}_D(T^i, T^j) = 2$. Note, that there is not exist $T^i, T^j \in S_D$ with $\mathrm{cr}_D(T^i, T^j) = 1$.

If there exists also such $T^k \in S_D$ that for every $T^l \in S_D$, $l \neq k$: $\mathrm{cr}_D(T^k, T^l) \geq 3$, we fixed $G \cup T^k$. And the same inequalities as in previous case (i) hold.

If there is not such $T^k \in S_D$, without lost of generality, let $\mathrm{cr}_D(T^{n-1}, T^n) = 2$ and let us fix $G \cup T^{n-1} \cup T^n$. In this step we are interested in all possible configurations of the subgraph F^i for some $T^i \in S_D$. Using cyclic permutation it is possible to prove $\mathrm{cr}_D(G \cup T^{n-1} \cup T^n, T^i) \geq 7$ for every T^i. So, we have

$$\mathrm{cr}(G + P_n) \geq \mathrm{cr}_D(K_{6,n-2}) + \mathrm{cr}_D(K_{6,n-2}, G \cup T^n \cup T^{n-1}) + \mathrm{cr}_D(G \cup T^n \cup T^{n-1}) \geq$$

$$\geq 6\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 7(n-2) + 3 > 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Thus, $n > s$. That implies $r \geq 1$. We assume that $T^n \in R_D$ with F^n having the only possible configuration. Every region of drawing of F^n, different to quadrangular region, contains exactly 2 vertices of G. The vertex t_j of T^j can not be placed in quadrangular region, because no edge of P_n is crossed. It implies $\mathrm{cr}(F^n, T^j) \geq 4$ for every T^j. Specially, if $T^j \in R_D$, then we apply the fact that $\mathrm{cr}(T^i, T^j) \geq 6$ if $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$.

We will discuss two possibilities over congruence n modulo 2.

- Let n be even. By fixing the graph F^n, we have

$$\begin{aligned}\operatorname{cr}(G+P_n) &\geq \operatorname{cr}_D(K_{6,n-1}) + \operatorname{cr}_D(K_{6,n-1}, G\cup T^n) + \operatorname{cr}_D(G\cup T^n) \geq \\ &\geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 6(r-1) + 4(n-r) + 0 = \\ &= 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 4n + 2r - 6 > 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + 2\Big\lfloor\frac{n}{2}\Big\rfloor.\end{aligned}$$

- Let n be odd. By fixing the subgraph T^n,

$$\begin{aligned}\operatorname{cr}(G+P_n) &\geq \operatorname{cr}_D(G+P_{n-1}) + \operatorname{cr}_D(G+P_{n-1}, T^n) \geq \\ &\geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 2\Big\lfloor\frac{n-1}{2}\Big\rfloor + 6(r-1) + 3s + 1(n-r-s) + 0 = \\ &= 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 2\Big\lfloor\frac{n-1}{2}\Big\rfloor + n + 2(2r+s) + r - 6 \geq \\ &\geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 2\Big\lfloor\frac{n-1}{2}\Big\rfloor + n + 2\Big(2n - 2\Big\lfloor\frac{n}{2}\Big\rfloor\Big) - 5 > \\ &> 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + 2\Big\lfloor\frac{n}{2}\Big\rfloor.\end{aligned}$$

Case 2: $\operatorname{cr}_D(G) = 1$.

It means that $r \geq 1$. Without lost of generality, we assume that $T^n \in R_D$. We can choose the vertex notation of the graph as shown in Figure 1(b). As there is the only drawing of the graph G with a crossing and there is also the only configuration of F^n, it is we get

$$\begin{aligned}\operatorname{cr}(G+P_n) &\geq \operatorname{cr}_D(K_{6,n-1}) + \operatorname{cr}_D(K_{6,n-1}, G\cup T^n) + \operatorname{cr}_D(G\cup T^n) \geq \\ &\geq 6\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 6(n-1) + 1 > 6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + 2\Big\lfloor\frac{n}{2}\Big\rfloor.\end{aligned}$$

Case 3: $\operatorname{cr}_D(G) \geq 2$.

There are several drawings of G with at least two crossings among its edges. Without lost of generality, we assume that $T^n \in R_D$. For every drawing of G with arbitrary vertex notation, there is a configuration of F^n uniquelly determined. We can apply the same idea as in previous case for all possible drawings of the graph G.

Thus, in every case, the crossing number of the graph $G + P_n$ is at least $6\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + 2\Big\lfloor\frac{n}{2}\Big\rfloor + 1$. It completes the proof.

2.1 Citation

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Normalized Interval Vectors are Divided into Two Classes in View of Belief Functions

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1 Introduction

Normalized interval vectors [1, 2] are often used in decision analysis because they show imprecise probabilities and imprecise weights under the condition that the sum of components is the unity (see [3, 4]). They are different from the sets of normalized vectors obtained from the conventional interval vectors through normalization. Their information about probabilities or weights is more imprecise than that of belief functions. In this paper, we show that a normalized interval vector is seen as a set of belief functions [5, 6, 7]. By the sum of center values of the normalized interval vector, we obtain two kinds of belief function sets: a set of normalized belief functions and a set of variant belief functions with a special condition.

2 Normalized Interval Vectors & Belief Functions

An interval vector $\boldsymbol{W} = (W_1, W_2, \ldots, W_n)$ [1, 2] is called a normalized interval vector if and only if it satisfies

$$w_i^{\mathrm{R}} \geq w_i^{\mathrm{L}} \geq 0,\ w_i^{\mathrm{R}} + \sum_{j \in N \setminus i} w_j^{\mathrm{L}} \leq 1,\ w_i^{\mathrm{L}} + \sum_{j \in N \setminus i} w_j^{\mathrm{R}} \geq 1,\ i \in N, \tag{1}$$

where $N = \{1, 2, \ldots, n\}$, $W_i = [w_i^{\mathrm{L}}, w_i^{\mathrm{R}}]$, $i \in N$.

A belief function $Bel : 2^N \to [0, 1]$ is characterized by a basic probability assignment (BPA) $m : 2^N \to [0, 1]$ satisfying the following properties (see [5, 6, 7]):

$$m(A) \in [0, 1],\ A \in 2^N,\ \sum_{A \in 2^N} m(A) = 1. \tag{2}$$

We consider a special m satsifying $m(\emptyset) = \max(0, 2-\sum_{i\in N}(m(\{i\})+\sum_{A\ni i} m(A))$. The belief function Bel and plausibility function $Pl : 2^N \to [0,1]$ are defined by

$$\begin{aligned} Bel(A) &= \sum_{B\subseteq A} m(B) - \min\left(m(\emptyset), \sum_{|B\setminus A|\geq 2} (|B\setminus A|-1)m(B)\right), \\ Pl(A) &= \sum_{B\cap A\neq\emptyset} m(B) + \min\left(m(\emptyset), \sum_{i\in A}\sum_{B\ni i,\ |B|\geq 2} m(B) - \sum_{B\cap A\neq\emptyset,\ |B|\geq 2} m(B)\right). \end{aligned} \tag{3}$$

$Bel(A)$ shows the lower probability of event A while $Pl(A)$ show the upper probability of event A. The treatment of $m(\emptyset) > 0$ is special in this model: we assign an additional probability mass $m^+(i|B)$ to each $i \in B$ and $|B| \geq 2$ from $m(\emptyset)$ as far as $m^+(i|B)$ satisfy $0 \leq m^+(i|B) \leq m(B)$, $i \in B$ and $|B| \geq 2$, $\sum_{i\in B} m^+(i|B) \leq (|B|-1)m(B)$, $|B| \geq 2$ and $\sum_{i\in N}\sum_{|B|\geq 2} m^+(i|B) = m(\emptyset)$.

The BPA m with $m(\emptyset) = 2-\sum_{i\in N}(m(\{i\})+\sum_{A\ni i} m(A) \geq 0$ can be obtained from a BPA m' with $m'(\emptyset) = 0$. Namely, BPA m is obtained by

$$\begin{aligned} &m(\{o_i\}) = m'(\{o_i\}) \text{ for } o_i \in \Omega, \\ &m(A) = \frac{1}{|A|-1} m'(A) \text{ for } A \subseteq \Omega \text{ such that } |A| \geq 2 \text{ and} \\ &m(\emptyset) = \sum_{A\subseteq\Omega:\ |A|\geq 3} \frac{|A|-2}{|A|-1} m'(A). \end{aligned} \tag{4}$$

On the other hand, from m with $m(\emptyset) = 2-\sum_{i\in N}(m(\{i\})+\sum_{A\ni i} m(A) \geq 0$, the corresponding BPA m' with $m'(\emptyset) = 0$ is obtained by

$$\begin{aligned} &m'(\{o_i\}) = m(\{o_i\}) \text{ for } o_i \in \Omega, \\ &m'(A) = (|A|-1)m(A) \text{ for } A \subseteq \Omega \text{ such that } |A| \geq 2 \text{ and } m'(\emptyset) = 0. \end{aligned} \tag{5}$$

Therefore, we have a one to one corespondence between a BPA m with $m(\emptyset) = 2-\sum_{i\in N}(m(\{i\})+\sum_{A\ni i} m(A) \geq 0$ and a BPA m' with $m'(\emptyset) = 0$. When $2-\sum_{i\in N}(m(\{i\})+\sum_{A\ni i} m(A) = 0$, BPA m is a fixed point in this correspondence.

3 Main Results

We obtain the following theorems.

Theorem 1. Let $\boldsymbol{W}$ be a normalized interval vector satisfying $\sum_{i\in N}(w_i^{\mathrm{L}}+w_i^{\mathrm{R}}) \geq 2$, there exists a BPA $m : 2^N \to [0,1]$ such that

$$m(\emptyset) = 0,\ Bel(\{i\}) = w_i^{\mathrm{L}},\ Pl(\{i\}) = w_i^{\mathrm{R}},\ i \in N. \tag{6}$$

Theorem 2. Let $\boldsymbol{W}$ be a normalized interval vector satisfying $\sum_{i\in N}(w_i^{\mathrm{L}}+w_i^{\mathrm{R}}) \leq 2$, there exists a BPA $m : 2^N \to [0,1]$ such that

$$m(\emptyset) = 2-\sum_{i\in N}(w_i^{\mathrm{L}}+w_i^{\mathrm{R}}) \geq 0,\ Bel(\{i\}) = w_i^{\mathrm{L}},\ Pl(\{i\}) = w_i^{\mathrm{R}}. \tag{7}$$

Theorem 3. Let $m : 2^N \to [0,1]$ be a BPA with $m(\emptyset) = 0$. Then, the interval vector $\boldsymbol{W} = (W_1, W_2, \ldots, W_n)$ defined by $W_i = [w_i^{\mathrm{L}}, w_i^{\mathrm{R}}] = [Bel(\{i\}), Pl(\{i\})]$, $i \in N$ is a normalized interval vector satisfying $\sum_{i\in N}(w_i^{\mathrm{L}} + w_i^{\mathrm{R}}) \geq 2$.

Theorem 4. Let $m : 2^N \to [0,1]$ be a BPA satsifying $m(\emptyset) = 2 - \sum_{i\in N}(Bel(\{i\}) + Pl(\{i\})) \geq 0$. Then, the interval vector $\boldsymbol{W} = (W_1, W_2, \ldots, W_n)$ defined by $W_i = [w_i^{\mathrm{L}}, w_i^{\mathrm{R}}] = [Bell(\{i\}), Pl(\{i\})]$, $i \in N$ is a normalized interval vector satisfying $\sum_{i\in N}(w_i^{\mathrm{L}} + w_i^{\mathrm{R}}) \leq 2$.

Theorem 5. Let $\boldsymbol{W} = (W_1, W_2, \ldots, W_n)^{\mathrm{T}}$ be a normalized interval vector satsifying $\sum_{i\in N}(w_i^{\mathrm{L}} + w_i^{\mathrm{R}}) \geq 2$. For any vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ satisfying $w_i \in W_i$ and $\sum_{i\in N} w_i = 1$, there exists a BPA $m : 2^N \to [0,1]$ satisfying $m(\emptyset) = 0$, $\sum_{i\in B} m_i(B) = m(B),\ B \subseteq N,\ \sum_{B\ni i} m_i(B) = w_i,\ i \in N,\ m_i(B) \geq 0,\ i \in N,\ B \subseteq N$ and $W_i = [w_i^{\mathrm{L}}, w_i^{\mathrm{R}}] = [Bel(\{i\}), Pl(\{i\})]$, $i \in N$.

Theorem 6. Let $\boldsymbol{W} = (W_1, W_2, \ldots, W_n)^{\mathrm{T}}$ be a normalized interval vector satsifying $\sum_{i\in N}(w_i^{\mathrm{L}} + w_i^{\mathrm{R}}) \leq 2$. For any vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ satisfying $w_i \in W_i$ and $\sum_{i\in N} w_i = 1$, there exists a BPA $m : 2^N \to [0,1]$ satisfying $m(\emptyset) = 2 - \sum_{i\in N}(w_i^{\mathrm{L}} + w_i^{\mathrm{R}})$, $\sum_{i\in B} m_i(B) = m(B),\ B \subseteq N,\ \sum_{B\ni i} m_i(B) = w_i,\ i \in N$, $m_i(B) \geq 0$, $i \in N$, $B \subseteq N$ and $W_i = [w_i^{\mathrm{L}}, w_i^{\mathrm{R}}] = [Bel(\{i\}), Pl(\{i\})]$, $i \in N$.

4 Concluding Remarks

In this paper, we investigated the relations between normalized interval vectors and belief functions. It was shown that a normalized interval vector corresponds to a set of belief functions. Corresponding to normalized interval vectors, two kinds of belief functions should be considered: one is normalized one, i.e., belief functions defined by a BPA m with $m(\emptyset) = 0$ and the other is a variant, i.e., belief functions defined by a BPA m with $m(\emptyset) = 2 - \sum_{i\in N}(m(\{i\}) + \sum_{A\ni i} m(A) \geq 0$. We found a one to one correspondence between a normal belief function and a variant belief function. As the result, a normalized interval vector is understood as a set of normalized and variant belief functions whose belief and plausibility function values of singleton with i-th elementary event are lower and upper bounds of the i-th interval of the normalized interval vector, respectively. The normalized condition in both normal and variant belief functions is well established. Therefore, the results obtained in this paper support the appropriateness of the normalized condition of normalized interval vectors.

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Preliminary Results from Experiments on the Behavior under Ambiguity

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Abstract

In the literature, some experiments proving that human decision-makers manifest an ambiguity aversion are described. In our knowledge, no one has studied a possibility to measure the strength of this aversion and its stability in time. The research, we have recently started to realize should find out answers to these and similar questions. The goal of this paper is to present some preliminary results to initiate a discussion that should help us to modify either the process of data collection and/or the analysis of the collected data.

1 Introduction

One of the goals of the research project GAČR 19-06569S is to find out how to construct normative models manifesting the same ambiguity aversion as human decision-makers. This term is used when speaking about the behavior, which is irrational if "rationality" means the behavior in agreement with the Savage's postulates formulated in his famous book [10]. The term is connected with the fact that human decision-makers do not like ignorance; they usually prefer uncertainty connected with a random experiment to total ignorance. The difference will be clear when describing the lotteries, which we use to test the behavior of experimental persons.

One of the first authors who experimentally studied this phenomenon was Ellsberg [4], and so it is not surprising that the behavior is often connected with the term *Ellsberg's paradox*. His experiments were often repeated [1, 5] but, in our knowledge, nobody made the experiments to measure the strength of ambiguity aversion. And this is why, during the first year of the above-mentioned project, we realize several experiments, the results of which should help us to characterize the concept of subjective ambiguity aversion. We want to find out to what extent we can rely upon our starting assumptions:

- In analogy to risk aversion, the ambiguity aversion is also a personal characteristic; not all decision-makers are influenced by this phenomenon in the same way.
- To some extent, it is possible to measure the strength of ambiguity aversion for individual human decision-makers.

Nevertheless, even if the above-stated assumptions are not declined, currently nobody knows to what extent the strength of the ambiguity aversion of a decision-maker depends also on the type of a decision task, and to what extent it is stable in time. All these are the open questions we are planning to study within the project mentioned above. As the starting point of our experimental research, we have designed the experiments, in which volunteers are asked to describe their behavior in several situations.

All the considered situations are formulated in the form of lotteries, in which the participants have a chance to win 100 CZK. At each of the situation, the content of a lottery drum is partially described, and the participants are asked to decide how much they are maximally willing to pay to be allowed to take part in the specified lottery. Details from the organization of these experiments are described in another paper presented at this conference. Here we just say that the following 14 situations are presented to experimental persons.

F1 The drawing urn contains 30 balls, five of each of the following colors: red, black, yellow, white, green, and azure. How much are you maximally willing to pay to take part in the lottery in which you win 100 CZK if the randomly drawn ball is red?

F2 The drawing urn contains 30 balls, five of each of the following colors: red, black, yellow, white, green, and azure. How much are you maximally willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is of the color of your choice?

I1 The drawing urn contains 30 balls, they may be of the following colors: red, black, yellow, white, green, and azure. You know nothing more, you even do not know how much colors are present in the urn. How much are you maximally willing to pay to take part in the lottery in which you win 100 CZK if the randomly drawn ball is red?

I1 The drawing urn contains 30 balls, they may be of the following colors: red, black, yellow, white, green, and azure. You know nothing more, you even do not know how much colors are present in the urn. How much are you maximally willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is of the color of your choice?

Rn This represents 8 lotteries for n= 5, 6, 7, ..., 12. The drawing urn contains n balls, each of which is either red, or black, or yellow, or white, or green, or

azure. You know that one and only one of them is red, nothing more. You even do not know how many colors are present in the urn. How much are you maximally willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is of the color of your choice?

E1 The drawing urn contains 15 red, black and yellow balls, you know that exactly 5 of them are red, you do not know the proportion of the remaining black and yellow balls. How much are you maximally willing to pay to take part in the lottery in which you choose a color and get 100 CZK if the randomly drawn ball has the color of your choice?

E2 The drawing urn contains 15 red, black and yellow balls, you know that exactly 5 of them are red, you do not know the proportion of the remaining black and yellow balls. How much are you maximally willing to pay to take part at the lottery in which you choose a color and get 100 CZK if the randomly drawn ball is either yellow or of the color of your choice?

2 Uncertain Knowledge representation

Considering the situations **F1** and **F2**, the knowledge can fully be described by a uniform probability distribution. Denoting the corresponding state space (i.e., a set of possible outcomes of a random draw) $\Omega = \{red, black, white, yellow, green, azure\}$ ($\Omega = \{r, b, w, y, g, a\}$ for short), for the uniform probability distribution $P_u(r) = P_u(b) =, \ldots, P_u(a) = \frac{1}{6}$. Notice that, due to additivity of probabilities, we also know (for example) that $P_u(\{r, g\}) = \frac{1}{3}$, and $P_u(\{b, y, a\}) = \frac{1}{2}$. Generally, for $\mathbf{a} \subseteq \Omega$, $P_u(\mathbf{a}) = \frac{|\mathbf{a}|}{6}$. It is also clear that from the situations introduced in the previous section, only the situations **F1** and **F2** can fully be described by probability distributions. For the description of the remaining situations we have to use another theoretical instrument.

2.1 Belief Functions

Consider the situation **Rn** describing One-red-ball example with n balls in a drawing drum. In this case we know only the probability $P_{\varrho,n}(r) = \frac{1}{n}$. We do not know the probabilities of other colors. But, again thanks to additivity of probability, we know that $P_{\varrho,n}(\{b, w, y, g, a\}) = P_{\varrho,n}(\Omega \setminus \{r\}) = 1 - \frac{1}{n}$. And this is the information that can be used to define a belief function. It is the information, which allows us to define the basic notion from this theory, so called *basic probability assignment*.

Since there is abundant literature on belief function theory (e.g., [11, 3, 12], and the papers introducing the models discussed in this paper [9, 8]), we presume that the reader is familiar with at least the foundations of this approach. Therefore, we introduce just the notation used in this paper.

The fundamental notion is that of a basic probability assignment (bpa), which describes all the information about the considered situation at our disposal. It is a function[1] $m : 2^\Omega \to [0,1]$, such that $\sum_{\mathbf{a}\in 2^\Omega} m(\mathbf{a}) = 1$ and $m(\emptyset) = 0$.

For bpa m, $\mathbf{a} \in 2^\Omega$ is said to be a *focal element* of m if $m(\mathbf{a}) > 0$. In what follows we will consider the following two special classes of bpa's representing the extreme situations:

- m is said to be *vacuous* if $m(\Omega) = 1$, i.e., m has only one focal element, Ω. A vacuous bpa is denoted by m_ι. It represents total ignorance, i.e., it represents the situations **I1** and **I2**.

- m is said to be *Bayesian*, if all its focal elements are singletons, i.e., for Bayesian bpa m, $m(\mathbf{a}) > 0$ implies $|\mathbf{a}| = 1$. Bayesian bpa's represent exactly the same knowledge as probability functions. As all focal elements of a Bayesian bpa m are singletons, we can define probability distribution P_m for Ω such that
$$P_m(x) = m(\{x\}) \tag{1}$$
for all $x \in \Omega$. Thus, Bayesian bpa's represent in our examples situations **F1** and **F2**.

Exactly the same knowledge that is expressed by a bpa m can also be expressed by a *belief function*, and by *plausibility function*.

$$Bel_m(\mathbf{a}) = \sum_{\mathbf{b}\in 2^\Omega:\, \mathbf{b}\subseteq \mathbf{a}} m(\mathbf{b}). \tag{2}$$

$$Pl_m(\mathbf{a}) = \sum_{\mathbf{b}\in 2^\Omega:\, \mathbf{b}\cap \mathbf{a}\neq \emptyset} m(\mathbf{b}). \tag{3}$$

In this paper we take advantage of the fact that for each bpa there exists a *credal set*, which is a convex set of probability distributions P on Ω defined as follows ($\mathcal{P}$ denotes the set of all probability distributions on Ω):

$$\mathcal{P}(m) = \left\{ P \in \mathcal{P} : \sum_{x\in \mathsf{a}} P(x) \geq Bel_m(\mathsf{a}) \text{ for } \forall \mathsf{a} \in 2^\Omega \right\}.$$

Notice that P_m defined by Equation (1) for a Bayesian bpa m is such that $\mathcal{P}(m) = \{P_m\}$, and that $\mathcal{P}(m_\iota) = \mathcal{P}$. From Equations (2) and (3), it can easily be deduced that for all $P \in \mathcal{P}(m)$
$$Bel_m(\mathsf{a}) \leq P(\mathsf{a}) \leq Pl_m(\mathsf{a}),$$
for all $\mathsf{a} \in 2^\Omega$. Thus, if $Bel(\mathbf{a}) = Pl(\mathbf{a})$ then we are sure that the probability of $\mathbf{a}$ equals $Bel(\mathbf{a})$. Otherwise, the larger the difference $Pl(\mathbf{a}) - Bel(\mathbf{a})$, the more uncertain we are about the value of the probability of $\mathbf{a}$. Using the terminology

[1] As usually, 2^Ω denote a set of all subsets of Ω.

of Srivastava [15], the greater this difference, the more ambiguity one has for the event (set of states) **a**.

The last notion we need in this paper is that of a famous *pignistic transform*, which was introduced in [16] and for decision making strongly advocated by Philippe Smets [13, 14]):

$$Bet\_P_m(x) = \sum_{\mathsf{a}\in 2^\Omega : x\in\mathsf{a}} \frac{m(\mathsf{a})}{|\mathsf{a}|}. \tag{4}$$

Notice, it defines for each bpa m a probability distribution, which is from the corresponding credal set $\mathcal{P}(m)$.

2.2 Measuring Strength of Ambiguity

The proposed way of measuring the strength of individual ambiguity aversion is based on the following mental model.

Consider situations **I1** and **F1** (or equivalently **I2** and **F2**). Usually (and it is confirmed also in our experiments) people are willing to pay more to take part in the lottery **F1** than in the lottery **I1**. This well known, seemingly paradoxical phenomenon, can hardly be explained by different subjective utility functions or by different subjective probability distributions. To explain this fact, we accepted a hypothesis that humans do not use their personal probability distributions but just *capacity functions* that do not sum up to one [6, 17]. Roughly speaking, the subjective probability of drawing a red ball is $\frac{1}{6}$ in the case that the person knows that the number of balls of all colors are the same in the drum. However, the respective "subjective probability" in the case of lack of knowledge is $\varepsilon < \frac{1}{6}$. *The lack of knowledge psychologically decreases the subjective chance of drawing the selected color – it decreases the subjective chance of success.* Thus, while we can accept that in situation **F2** the decision-maker considers that the probabilities of individual colors are $\frac{1}{6}, \frac{1}{6}, \ldots, \frac{1}{6}$, in situation **I2** these "subjective probabilities" are only $\varepsilon, \varepsilon, \ldots, \varepsilon$. Assuming this decrease is linear with the subjective strength of ambiguity, we can measure it by a personal coefficient of ambiguity α, which can be expressed, in case that the person is willing to pay a CZK in situation **F1** and b CZK in situation **I1**, by the following simple formula

$$\alpha = \frac{a-b}{a}. \tag{5}$$

The higher this coefficient, the stronger the aversion. Namely, if the person is willing to pay a CZK when her expected probability of success is $\frac{1}{6}$ (situation **F1**), then, in case of the decreased probability of success, which is $(1-\alpha)\cdot\frac{1}{6}$ (in case of **I1**), she is willing to pay

$$(1-\alpha)\cdot a = (1-\frac{a-b}{a})\cdot a = a-(a-b) = b.$$

Let us, now, show how this personal coefficient of ambiguity influences behavior of an experimental decision maker in situations **Rn**.

As we have already said at the beginning of Section 2.1, in situation **Rn**, the content of the drawing drum is described by bpa $m_{\varrho,n}$ given as follows:

$$m_{\varrho,n}(\mathbf{a}) = \begin{cases} \frac{1}{n}, & \text{if } \mathbf{a} = \{r\}; \\ \frac{n-1}{n}, & \text{if } \mathbf{a} = \{b,\ g,\ o,\ y,\ w\}; \\ 0, & \text{otherwise,} \end{cases}$$

and the corresponding belief function is $Bel_{m_{\varrho,n}}(\{x\}) = 0$ for all $x \in \{b,\ g,\ o,\ y,\ w\}$, and $Bel_{m_{\varrho,n}}(\{r\}) = \frac{1}{n}$.

For the sake of simplicity let us accept here the Smets' advice [14] saying that for decision making one should compute the expected value using the pignistic transform (for a survey of other probabilistic transforms see [2], and for more discussion on the problem of a probabilistic transfrom selection see [7]), which is

$$Bet\_P_{m_{\varrho,n}}(x) = \begin{cases} \frac{1}{n}, & \text{if } x = r; \\ \frac{n-1}{5n}, & \text{for } x \in \{b,\ g,\ o,\ y,\ w\}. \end{cases}$$

If there were not for the ambiguity, we should use it directly for the computation of the expected winnings. However, in our approach, we have to decrease it using the personal coefficient of ambiguity aversion α. We have to decrease it at each point of Ω proportionally to the strength of the ambiguity connected with the considered point. Realize, that $Bet\_P_{m_{\varrho,n}}(x) - Bel_{m_{\varrho,n}}(\{x\}) \geq 0$, and equals 0 if and only if $Pl_{m_{\varrho,n}}(x) = Bel_{m_{\varrho,n}}(\{x\})$. Thus, if $Bet\_P_{m_{\varrho,n}}(x) = Bel_{m_{\varrho,n}}(\{x\})$, we know the respective probability exactly, we do not have any ambiguity about its value. However, the greater the difference $Bet\_P_{m_{\varrho,n}}(x) - Bel_{m_{\varrho,n}}(\{x\})$, the greater the ambiguity, and therefore we have to decrease probabilities $Bet\_P_{m_{\varrho,n}}(x)$ accordingly. After the decrease, they do not sum up to one, any more, and therefore we call them *reduced weights*, and compute them according to the following formula:

$$r_{m,\alpha}(x) = (1-\alpha)Bet\_P_m(x) + \alpha Bel_m(\{x\}). \tag{6}$$

Thus, in situation **Rn** we get:

$$r_{m_{\varrho,n},\alpha}(x) = \begin{cases} \frac{1}{n}, & \text{if } x = r; \\ (1-\alpha)\cdot\frac{n-1}{5n}, & \text{for } x \in \{b,\ g,\ o,\ y,\ w\}. \end{cases}$$

Considering (for the sake of simplicity just two) gain functions $g^r(x)$, and $g^w(x)$ (corresponding to betting on red and white color, respectively), the total subjective rewards are as follows. When betting on red it equals

$$R_{m_{\varrho,n},\alpha}(r) = \frac{1}{n}g^r(r) + \sum_{x\in\Omega: x\neq r} \frac{(1-\alpha)(n-1)}{5n} g^r(x) = \frac{100}{n},$$

Table 1: One Red Ball Example: Total subjective reward as a function of the coefficient of ambiguity aversion α, and the number of balls n.

		$R_{m_{\varrho,n},\alpha}(\mathrm{w})$						
n	$R_{m_{\varrho,n},\alpha}(\mathrm{r})$	α=0	α=0.1	α=0.2	α=0.3	α=0.4	α=0.5	α=0.6
5	20.00	16.00	14.40	12.80	11.20	9.60	8.00	6.40
6	16.67	16.67	15.00	13.33	11.67	10.00	8.33	6.67
7	14.29	17.14	15.43	13.71	12.00	10.29	8.57	6.86
8	12.50	17.50	15.75	14.00	12.25	10.50	8.75	7.00
9	11.11	17.78	16.00	14.22	12.44	10.67	8.89	7.11
10	10.00	18.00	16.20	14.40	12.60	10.80	9.00	7.20
11	9.09	18.18	16.36	14.55	12.73	10.91	9.09	7.27
12	8.33	18.33	16.50	14.67	12.83	11.00	9.17	7.33

and analogously, for betting on white

$$R_{m_{\varrho,n},\alpha}(\mathrm{w}) \quad = \quad \frac{1}{n} g^{\mathrm{w}}(\mathrm{r}) + \sum_{x \in \Omega : x \neq \mathrm{r}} \frac{(1-\alpha)(n-1)}{5n} g^{\mathrm{w}}(x) = \frac{100(1-\alpha)(n-1)}{5n}.$$

Some of the values of these functions are tabulated in Table 1. From this table we see that, for example, a person with $\alpha = 0.4$ should bet on red color for $n \leq 9$, because for these $R_{m_{\varrho,n},\alpha}(\mathrm{r}) > R_{m_{\varrho,n},\alpha}(x)$ $(x \neq \mathrm{r})$, and bet on any other color for $n \geq 10$, because for these n, $R_{m_{\varrho,n},\alpha}(\mathrm{r}) \leq R_{m_{\varrho,n},\alpha}(x)$ $(x \neq \mathrm{r})$. This means that for $n \leq 9$, it is subjectively more advantageous to bet on the red color. In the next section we say that the *computed breaking point* of such a person is 10.

We conclude this section mentioning that the description of the reduced function for the Ellsberg's examples is more complicated, because the gain function for the situation **E2** equals 100 for two values. In this case, we have to consider both pignistic transform and reduced weights functions as mappings on 2^{Ω}. Since we do not necessarily need it in the rest of this paper, we do not describe it here and refer the interested reader to [9, 8].

3 Results from Experiments

At the time of preparation of this paper, we have data from 49 respondents. Naturally, not all the respondents undertook the task with the same responsibility. It can be seen, among others, from the time, which they needed for finishing the task. In average, the respondents needed 5 minutes, and 19 seconds, but two of them finished the whole task in less then one minute (33 and 36 seconds). A similarly irresponsible attitude may be expected from the respondents who were willing to bet just one CZK (or 0 CZK) for all twelve situations. Naturally, for correct statistical data processing we should clean the data, and delete these obviously misleading responses. Because we do not have enough data and we do not have criteria how

to detect misleading data, for this preliminary discussion we keep all the data as they were collected.

3.1 First Glance Comments

The reader certainly noticed that in situations **F1** and **F2** (in the same way as in situations **I1** and **I2**) the participants have the same information about the content of the drawing drum. The difference is just that in **F1** the winning color is predetermined (red), while in **F2** the participant determines the winning color herself. We included both of them into the battery of the considered situations, because we were not sure whether the participants would not suspect the organizers to exclude red color from the drawing drum in case that the winning color (red) is predetermined. This suspicious appeared false. The total amount of money bet in **I1** was 292 CZK, while in **I2** they altogether bet 290 CZK (for **F1** and **F2** the total amounts were 584 and 557 CZK, respectively).

The only observation, which surprised us at the first glance, concerns the behavior of the respondents in situations **E1** and **E2**. The reader familiar with the famous Elsberg's paper [4] already noticed that these situations were designed to repeat the Eslberg's experiment. Let us briefly recollect his example.

In [4] (pp. 653–654), Ellesberg considers the situation with a drawing drum containing 30 red balls and 60 black or yellow balls, the latter in unknown proportion. With this drum, Ellsberg considers two experiments. The first experiment (which we repeat as **E1** in our study) finds out whether people prefer betting on red or black ball, in the case they get the reward if the ball of the respective color is drawn at random. According to his observations, "very frequent pattern of response is that betting on red is preferred to betting on black". This corresponds also with our results, in which 36 (out of all 49) respondents bet on black color. In the second Ellsberg's experiment (simulated in our experiments as **E2**), a decision-maker can bet on red and yellow, or on black and yellow. Again, the participant gets the reward in case that the randomly drawn ball is of one of the selected colors. In this case, the Ellsberg's observation is that " betting on black and yellow is preferred to betting on red and yellow", which is not in the agreement with the results we have achieved. In our case, only 16 participants betted on black color.

3.2 Coefficient of Ambiguity α

Let us turn our attention to what can be said about the coefficient of ambiguity on the basis of the considered preliminary data. As a starting point, we computed this coefficient according to Formula (5) for all respondents. Having two pairs of situations, we computed two such coefficients; one from the bets in situations **F1** and **I1**, the other from bets in situations **F2** and **I2**. The situations are submitted to the participants in a random order, so it is quite interesting to what extent the two coefficients differ from each other. The results are depicted in Figure 1. Each point corresponds to one respondent (or several, if both coefficients coincide for

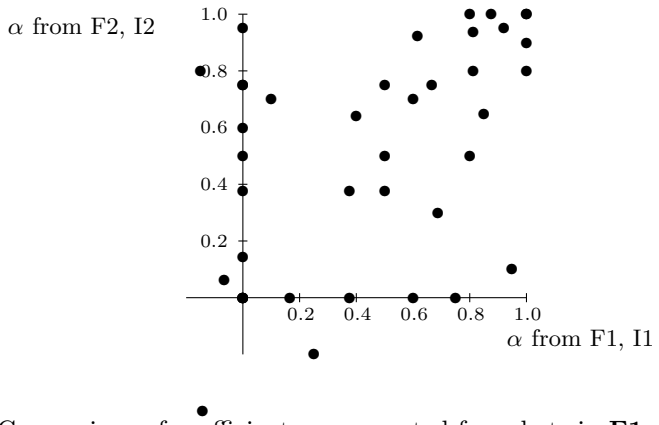

Figure 1: Comparison of coefficients α computed from bets in **F1**, **I1** and **F2**, **I2**.

several respondents), who took part in our experiments. The coordinates of each point are the respective coefficients α. From this figure, we see several unexpected facts. First, a few participants are exhibiting *ambiguity inclination*; their coefficient of ambiguity α is negative. Second, there is not a small part of participants, who manifest the ambiguity aversion just in one pair of situations (either in **F1**, **I1**, or in **F2**, **I2**) – see the points on the axes. For only a small number of participants, both coefficients are close to each other. Naturally, we have only a small amount of data (some of which should be removed because of the reasons mentioned above), so we cannot make any final conclusions. Therefore, in what follows, we consider just one coefficient of ambiguity, which is computed from sum betted together in **F1** and **F2**, and the sum betted in **I1** and **I2** together. To simplify the next exposition, let us call these coefficients the *joint coefficients of ambiguity*.

Going back to situations **Rn**, and assuming that the joint coefficient α expresses the strength of the ambiguity aversion of the individual respondents, we can, using Table 1, estimate the breaking point, i.e., the number of balls when the participants start betting on another color than the red one. We compute it for each experimental person using her personal joint coefficient of ambiguity. Comparing the computed breaking point with that, which can be read from data, we have found out that for half of the respondents (more precisely, for 25 out of 49) the breaking point computed from the models does not differ from the actual breaking point by more than one.

It is worth mentioning that from three respondents with negative joint coefficient of ambiguity, one did not bet on red color even for $n = 5$, and another betted on blue color already for $n = 6$. Thus, these two respondents displayed their ambiguity inclination even when reacting in situations **Rn**. Again, even this surprising result must be taken with a great care because of a small amount and not cleaned data.

Though the amount of money, the respondents are maximally willing to pay to take part in lotteries, is not in the center of our interest, the question is whether

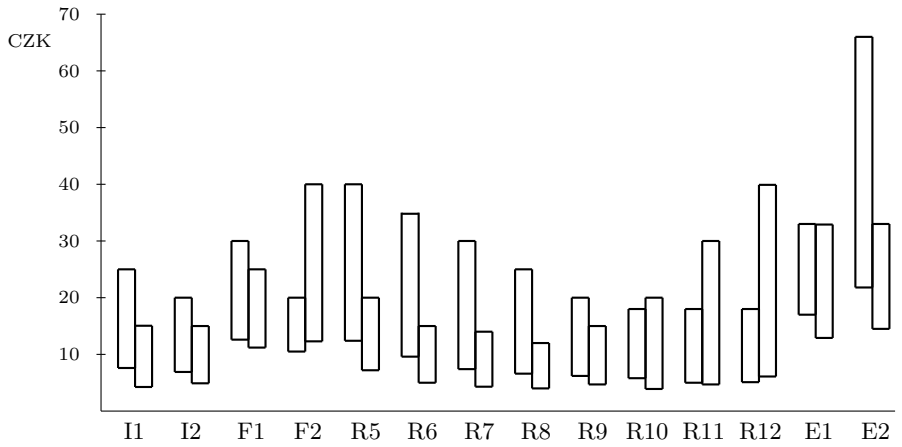

Figure 2: Maximal and average bets on individual lotteries.

these numbers should be taken into consideration when evaluating the quality of data. For example, is it meant seriously, if an experimental person claims that she is willing to pay $5, 8, 10, 15, 20, 30, 40$ CZK in situations **R5**, **R6**, ..., **R12**, respectively? Some irrational behavior of respondents can also be read from Figure 2, in which each situation (lottery) is described with two boxes. Left-hand box corresponds to those 25 respondents, for which the breaking point from lotteries R5 – R12 does not differ from that computed using the joint coefficient α by more than one. The right-hand box is computed from data of the rest of 24 respondents. The lower edge of each box shows the average of the amounts the respondents are willing to pay for taking part in the lottery, the upper edge shows the maximal value (it does not have the sense to depict the minimum because of the above-mentioned respondents stating that they are willing to pay just 1 CZK (or 0 CZK) in all situations).

4 Conclusions

In the paper, we described the experiments we are realizing to better understand the concept of individual subjective ambiguity aversion. The analysis of first data arises more questions than answers, and this is the main reason, why we present this paper at the Czech-Japan seminar. We want to initiate the discussion that should help us to find answers to the following questions:

- Is it possible to minimize the number of respondents replying the questions without thinking?
- Should the data be cleaned before their processing?

- If yes, what criteria should be used to clean the data?

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How to make the Olympics: Tournament planning strategy using fuzzy inference system

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Abstract

An introduction to the winning qualification strategy to the Olympics in badminton. The tournament planning strategy is based on many factors which you cannot, as a player, influence such as tournament status, level of players entering tournaments, results of the other players etc. Thus in this paper, we introduce a generalized framework of IF-THEN fuzzy rules for the tournament's planning strategy. The framework is based on the results of athletes who qualified for the latest Summer Olympic Games in Rio 2016.

1 Introduction

Planning is a crucial part of the sportsmen's life. A well planned season delivers success and improvement for the players. On the other hand a wrongly peaked season can negatively impact all the previous sportsman's results and preparations. During planning the key factor is to make the right decisions. Decision making in sport is constantly developing areas especially with technological development. Thus the scientific part of these decisions become increasingly important, because that could be the only competitive advantage a sportsman has in comparison with the others. Badminton has been a Summer Olympic Sport since 1992. Understanding how to choose the tournaments which could be a qualification for the Olympics is important for planning the season and pacing peak performance to achieve the necessary results at these tournaments. This fuzzy inference model allows coaches to visualise rules for planning tournaments based on nominated pairs.

2 Literature review

Planning is in fact the root of the training procedure during the period which is divided into smaller cycles. [1]. Planning is based on the concept of the periodisation. Such scheduling increases the training organisation and provides the possibility of order and scientifically conducting the periodisation [2]. There are many occasions for studying decision making in sport. The article published by [5] describes various situations where decisions are made. This article proposes the idea of immediately determining the performance by fans and the media. There are technical and tactical aspects which influence decision making.

The coaching style is one of the impacts as well. [4] highlights three characteristics of decision-making in the field of sports. The decisions are naturalistic which means that coaches and athletes naturally encounter the decision in a sporting environment with some degree of task familiarity. The majority of the decisions in the sport are dynamic despite some decisions being a lengthily process. The author places emphasis on decisions which are often made on-line during the tasks or intense timely stressful situations which it's related to, but distinct from, the dynamic nature of sporting decisions. Thus the element of variability must be realised when studying sporting decisions. The article published by [6] introduced a method to use fuzzy theory for annual planning. The plan is divided into sections (weekly cycles, tournaments, training camps, testing or review) and thus the sections are easily managed while planning the season. Despite badmintons long history in the world, there has not yet been any attempts to use fuzzy theory for planning the tournament or the season in badminton.

3 Sport strategic decision

Planning tournaments has always been tricky. It seems the best option is to enter a high number of tournaments, however this is not practical, because you need to achieve great results on each tournament and for that you need a lot of training. It is a tough question for the coach to plan a season to attain enough points and get a good ranking especially when only the last 52 weeks are counted in the BWF ranking system. Planning for the Olympics is in a four year cycle, but the key weeks are the 52 qualification weeks involving a lot of tournaments. Thus the question is which tournament to go for and which achievements have to be made to secure one of the 16 spots for men's/ women's doubles.

3.1 Qualification system for the Tokyo 2020

The qualification system is known before the qualification's cycle begins so the players and coaches have enough time to plan. Comparing the Rio 2016 and Tokyo 2020 qualification system there are some changes thus the coaches will need to use more science behind the planning qualification's cycle. The qualification's cycle starts on 29 April 2019 and ends on 26 April 2020. The World Ranking Lists of

30 April 2020 will be used to allocate the athletes places in Mens and Womens Singles, and Mens, Womens and Mixed Doubles. There are four main constrains the players/coaches have to be aware of. There is a total quota of 172 players for badminton including one women's/men's spot for the host country. There is a quota per NOC (National Olympic Committee). The NOC is allowed to have the 8 mens and 8 womens across all 5 events however the NOC can only put forward a maximum of two pairs if both are within the top 8 of the World Ranking list as of 30 April 2020. The last limitation is the maximum number of athletes per event. There could be only 38 players in women's/men's singles and 16 pairs in women's/men's/mixed doubles. Each event has to include at least one pair from all five BWF Continental Confederations (Pan America, Oceania, Europe, Africa, Asia), provided that the pairs are ranked within the top 50 on the cut-off date (30 April 2020).

The Badminton World Federation World (BWF) Ranking is the official ranking of the BWF for badminton players who participate in tournaments sanctioned by BWF. It is used to determine the qualification for the World Championships and Summer Olympic Games as well as BWF World Tour tournaments. The BWF World Ranking was introduced to determine the strength of the players. The points awarded are based on the final results of each tournament participated in for the past 52 weeks. The points awarded are based on the level and progress of the tournament from each player/pair. If a player or pair has participated in 10 or fewer World Ranking tournaments, then the ranking is worked out by adding the points won at the tournaments. If a player or pair has participated in 11 or more World Ranking tournaments, only the 10 highest points scored in the tournaments during the 52-week period count towards their ranking. The highest possible ranking points are 116,000.

3.2 Men's/ Women's double

Table 1 gives a summary of the ranking of each pair through the last 52 weeks. The majority of players are from Asia, but at first glance there is no clear guidance for the number of tournaments necessary for qualification.

4 IF-THEN rules approach

Using fuzzy logic allows us to build the fuzzy inference system. The case study used Matlab for building the fuzzy inference system based on the Mamdani logic using theory from [3]. IF-THEN rules for each pair were created using the data from the qualification for Rio 2016. Even if the qualification's system is slightly altered then still the most important rules remain. The system is based on the results of each tournament for 52 weeks. The model was created using the data from Rio 2016 as illustrated in figure 1.

For the qualification period there were more than 100 tournaments. The whole list of tournaments are listed on the BWF website. There are five events played

Table 1: Men's and Women's qualified pairs

Qualif. position	BWF Ranking		BWF National federation		Earned Points		Number of tournament	
	Men	Women	Men	Women	Men	Women	Men	Women
1	1	1	Korea	Japan	92,480	82,469	19	19
2	2	2	Indonesia	Indonesia	75,380	78,649	19	20
3	3	3	China	China	73,117	77,369	14	11
4	4	5	Korea	Denmark	71,867	74,604	26	16
5	5	6	China	Korea	70,915	72,097	19	23
6	7	7	Japan	China	66,327	71,523	18	15
7	8	8	Denmark	Korea	65,604	64,876	11	23
8	10	11	Russia	Netherlands	58,073	53,705	21	23
9	14	14	Malaysia	Indonesia	55,098	45,811	25	22
10	18	15	Taipei	Bulgaria	47,151	45,157	21	25
11	19	17	England	Thailand	45,754	44,336	22	24
12	20	21	Indonesia	Malaysia	44,549	42,067	28	19
13	27	22	Poland	Hong Kong	38,425	41,480	29	15
14	28	23	Germany	Germany	37,096	40,070	28	28
15	35	26	USA	England	31,867	37,540	29	30
16	46	29	Australia	USA	26,104	36,001	18	27

at the Olympics - men's single, women's single, men's double, women's double and mixed double. It is easier to qualify to the Olympics in the single events because 38 players were selected to compete compared with just 16 pairs in the double events. Thus the case study is focusing only on the men's and women's double because it' s harder to secure the Olympic place therefore the planning is more challenging. The rules reflect the qualification's criteria in particular the one stating that the 16 pairs in each doubles category must include pairs from all five BWF Continental Confederations (Asia, Europe, Oceania, Pan America, Africa), provided that the pairs are ranked within the top 50 by the cut-off date and the NOC has a maximum of two pairs that are both within the top 8 of the World Ranking list.

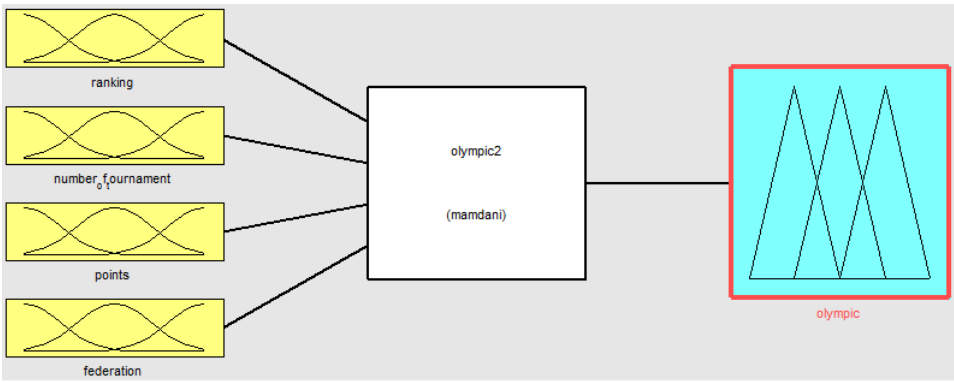

Figure 1: Fuzzy inference model

4.1 Modeling in MATLAB SIMULING

The system is based on the results in the different categories (men's and women's double) and the performance evaluation. The model for 16 pairs was created by the Fuzzy Logic Designer in Matlab. The crucial decision was to select the right model. There was either a model which included all tournaments during qualification or a model which included different variables. For the first model mentioned we have the data and therefore this model is more accurate albeit restricted. A benefit for the second model was that it uses more variables and easier to use and adjust. Thus the second model (figure 1) was created based on in-depth evaluation of both models.

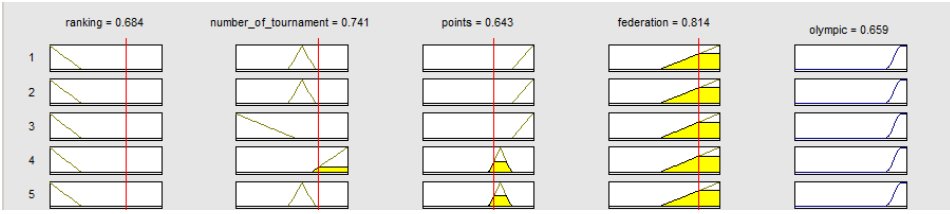

Figure 2: Rules define in fuzzy inference system

The variables in the rules are chosen based on their importance. The model includes four main variables - BWF ranking, number of tournaments, points reached, confederations involved in the fuzzy inference system. It is hard to define which BWF ranking the player will achieve and the fuzzy logic easily defines the ranking through the season. The number of tournaments is hard to determine particularly at the beginning of the qualification period. This model allows quick modification

of the number of planned tournaments. Points reached is defined between low, average, high and very high. The easiest variable to determine is confederation. The qualified men's and women's double pairs show similar results. Some of the sixteen rules are illustrated in the figure 2.

We can analyse the results from a different perspective using the surface viewer. Figure 3 shows the most significant surface view determined after comparing all surface analyses. As seen in figure 3 a different strategy is required for planning

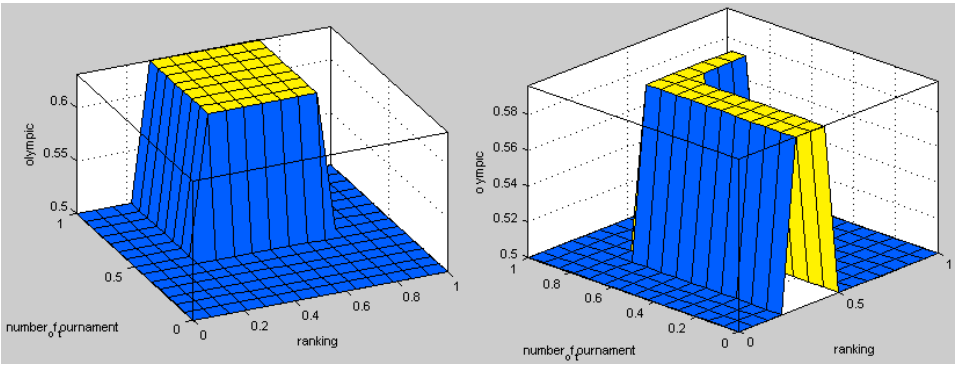

Figure 3: Men's and Women's double surface view for points and number of tournaments

men's and women's double tournaments. Surface view showed that in men's double there is a greater chance to qualify if a player enters more tournaments whereas in women's double there isn't the strong dependency between achieving the top qualification and the number of tournaments entered. We can see from the model that the same points are associated with the different qualification place and ranking. For men's double it is clear that there is a larger points range between the first and the last qualifier while for women's the points distribution is narrower.

Creating the strategy of planning tournaments using this system is easy to use because it allows dynamic changing. If a player gets injured during the season you can still adjust the strategy of choosing which tournaments to enter (the entry deadline is usually around a month before the tournament) and lets the player recover. You can work with the model and change the rules in real time. It enables accurate modelling of the opponent's situation and it gives the coach a realistic idea of situation. It helps coaches make more precise decisions based on the previous data as well on the real-time data.

5 Conclusion

The purpose of this article, based on the extensive review, was to identify the key variables for planning tournaments and creating a fuzzy inference model using the

MATLAB SIMULING software. The article also emphasises the different approach for women's and men's double.

However, the results brought significant improvement in choosing tournaments and then implementing those tournament choices in the sportsman's year planning, the model has some constraints which need to be considered. There are 52 weeks which is a long period to make precise predictions especially when multiple tournaments are being held on the same weekend. To build a more robust model more players need to be involved as well as all tournaments played by the successful player but not counting towards their ranking.

Acknowledgments

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On exact algorithm for shortest path network interdiction

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Abstract

The research of network interdiction (NI) concerned with networks each of which is to provide a certain kind of service, and its performance depends of its structure and the status of its links and/or nodes. Links and/or nodes of the network can be interdicted (i.e. destroyed, failed, deleted, etc.) so as to reduce network performance. In the literature, evolutionary algorithm to the shortest-path network interdiction problem has been developed for optimizing two objectives : (1) maximization of shortest-path length and (2) minimization of interdiction strategy cost. This approach allows approximating the optimal Pareto set of network interdiction strategies. This paper presents a new algorithm that allows getting optimal solutions for this problem. This method is based on graph theory to analyze strategies' shortest path and a branch and bound method according to the change of distance (from source node to each node) by interdiction of each edge. Shortest-path of the network is determined by Dijkstra's algorithm and the branch-and-bound procedure partially enumerates interdiction strategies. Examples for different sizes of networks and network behavior are used throughout the paper to illustrate and validate the approach.

1 Introduction

The research of network interdiction (NI) concerned with networks each of which is to provide a certain kind of service. The shortest path length, maximum flow and reliability are considered as evaluation values of the network function. The purpose of this problem is to optimize (maximize or minimize) the functions (that is, the evaluation values representing them) of network. This is achieved by "interdiction" - destroying a edge or increase or decrease the evaluation value associated with the edge - a certain edge of network. Identifying edges to be interdicted is the NI. This problem is applied to various fields, and various models exist. For example, military[2, 3], US drug issues[4, 5], infectious control[8], supply chain[13], evaluation of performance in the event of communication network failures[6, 7].

In general, it is assumed that cost is required to interdict edges. So, we understand that the shortest path length and interdiction cost are in a trade-off relation. In the literature, there are problems aimed at reducing the functions of the network as much as possible at a limited cost, and there are problems aimed at optimizing the functions and minimizing the interdiction cost without cost restrictions[11, 12, 14]. That is, the latter is a bi-objective optimization problem. We focus on the bi-objective shortest path network interdiction problem with the objective function of maximizing the shortest path length and minimizing the total interdiction cost. It is described in [9] that the shortest path NI is NP-complete. This problem is proposed in [10]. It is proposed a multi-objective evolutionary algorithm (MOEA) as a method for this problem[10]. This heuristic algorithm provides an approximation of the optimal Pareto set for this problem.

The existing algorithm[10] explore regions of the solution space based on probabilistic search based on a function value of solutions generated at each cycle to obtain high-quality solutions. While this algorithm can be implemented in a short time, there is a problem that the output is not guaranteed to be the optimal Pareto set because it is heuristic.

In this paper, we proposed an exact algorithm for this problem. In this algorithm, Solutions are searched based on a branch and bound method using Dijkstra's algorithm. Our approach of simultaneously performing branch and bound operation and calculating the shortest path length with Dijkstra's algorithm not only guarantees the quality of the output, but also reduces enormous computation time and memory.

In this paper, we demonstrate the usefulness of the proposed algorithm by implementing with several networks and comparing it with output obtained with the existing algorithm. We also consider the challenges of this approach and future work.

The rest of the paper is organized as follows. In Section 2, we describe the formulation of the bi-objective network interdiction problem, In Section 3, we propose our approach and describe in detail. In Section 4, we apply our algorithm to some networks, evaluate and illustrate the performance of it by comparing with the results obtained with existing algorithm. In Section 5, we present conclusions.

2 Problem Formulation

Let us formulate our problem, namely *the shortest path network interdiction problem*. An *instance* $I = (G, s, t, \alpha, \beta, c)$ of our problem is a 6-tuple such that

- $G = (N, E)$ is a directed graph (network) specified by a set N of nodes and a set $E \subseteq \{(i, j) \in N \times N \mid i \neq j\}$ of directed edges (links),
- $s \in N$ and $t \in N$ are specified nodes satisfying $s \neq t$, respectively called *source* and *sink* of G.
- $\alpha : E \to \mathbb{R}_{\geq 0}$ is a *nominal distance function* such that the nominal distance of each edge $e \in E$ is represented by $\alpha(e)$,
- $\beta : E \to \mathbb{R}_{\geq 0}$ is a *prolongation function* such that $\beta(e)$ represents how much the distance of edge $e \in E$ is prolonged due to its interdiction, i.e., the interdicted distance of each edge $e \in E$ is $\alpha(e) + \beta(e)$, and
- $c : E \to \mathbb{R}_{\geq 0}$ is a *cost function* such that $c(e)$ is the interdiction cost of edge $e \in E$.

Here, we assume that interdiction of each edge requires a certain amount of cost and it increases the nominal distance of the edge, and thus, those values are assumed to be non-negative. In fact, we shall assume that those values to be positive. If $\beta(u, v) = 0$ for some $(u, v) \in E$, such edges is not be interdicted by default. If $\beta(u, v) > 0$ and $c(u, v) = 0$ for some $(u, v) \in E$, we shall assume the such edge is to be interdicted by default.

A *solution* for an instance $I = (G, s, t, \alpha, \beta, c)$ is an interdiction strategy $\sigma \subseteq E$, which denotes the set of edges to be interdicted. Each interdiction strategy σ is evaluated by two measures, namely, the length of the shortest path from source s to sink t and the total interdiction cost when all edges in σ are interdicted. More formally, under an interdiction strategy σ, the length of shortest path from s to t can be expressed by $L(\sigma)$ defined as follows:

$$L(\sigma) = \min \left\{ \sum_{e \in P} \alpha(e) + \sum_{e \in P \cap \sigma} \beta(e) \;\middle|\; P \subseteq E \text{ is an } s\text{-}t \text{ path in } G \right\}.$$

The total interdiction cost $C(\sigma)$ is the total of interdiction costs of all edges in σ.

$$C(\sigma) = \sum_{e \in \sigma} c(e).$$

Hence, we are dealing with a bi-objective optimization such that $L(\sigma)$ is to be maximized and $C(\sigma)$ is to be minimized among all interdiction strategies.

$$\begin{aligned} \max \quad & L(\sigma) \\ \min \quad & C(\sigma) \\ \text{subject to} \quad & \sigma \subseteq E \end{aligned}$$

It is obvious that $L(\sigma)$ is maximized when all edges in E are interdicted (i.e., $\sigma = E$), and $C(\sigma)$ is minimized when no edges are interdicted (i.e., $\sigma = \emptyset$), and indeed, the two objectives are in a trade-off relation. Hence, we solve our problem by finding the set of all Pareto optimal interdiction strategies. Let $\sigma \subseteq E$ be an interdiction strategy.

- We say that $\sigma \subseteq E$ is *dominated* by another interdiction strategy $\sigma' \subseteq E$ if
  - both $L(\sigma) \leq L(\sigma')$ and $C(\sigma) \geq C(\sigma')$ are satisfied, and
  - at least one of $L(\sigma) < L(\sigma')$ and $C(\sigma) > C(\sigma')$ holds.
- We say that $\sigma \subseteq E$ is *Pareto optimal* if σ is not dominated by any other interdiction strategy.

3 A Branch and Bound Approach

In general, a sketch of a branch and bound approach to a multi-objective optimization problem is such that

- a partial solution p and a set P of non-dominated solutions are maintained all the time during the process,
- branches are made by deciding one by one the value of a decision variables,
- bounds of the objectives (for all complete solutions can be obtained by extending p) are estimated in order to eliminate unnecessarily branches, and
- once p becomes a complete solution, the set P of non-dominated solutions is to be updated by the set of all non-dominated solutions among $P \cup \{p\}$.

The keys are

- to find an effective ordering of decision variables (and their values as well), and
- to estimate bounds of our objectives in efficient ways.

These keys are strongly related to each other.

3.1 Dijkstra's Algorithm based Edge Selection

For our problem, each of the decision variables corresponds to an edge and decides whether such an edge is to be interdicted or not. Hence, in our branch and bound process, a partial solution is represented by a pair (Y, σ) of a set $Y \subseteq E$ of decided edges and a set $\sigma \subseteq Y$ of interdicted edges among those in Y (i.e., σ is an interdiction strategy). At each branching stage of the process, based on the current partial solution (Y, σ), an edge $e \in E \setminus Y$ is selected and the process may

make at most two branches respectively with partial solutions $(Y \cup \{e\}, \sigma)$ and $(Y \cup \{e\}, \sigma \cup \{e\})$. Without removing any branch, the process becomes an exhaustive search with any ordering of edge selection. However, some ordering may help to estimated bounds of our objectives in some efficient ways, and hence, may help to eliminate some unnecessary branches. We find out that Dijkstra's algorithm provides us an answer.

Dijkstra's algorithm solves the (single-source) shortest path problem when every edge (in the directed graph under consideration) has non-negative length. More precisely, the algorithm

- receives a directed graph $G = (N, E)$, a weight function $w : E \to \mathbb{R}_{\geq 0}$, a source node $s \in N$ as input, and
- returns, for each $u \in N$, the length $\textsc{dist}(u)$ of a shortest path from s to u in G.

Algorithm 1: Dijkstra's Algorithm

```
1 d(s) ← 0
2 for u ∈ N \ {s} do
3 |  d(u) ← ∞
4 X ← {s}
5 while X ≠ N do
6 |  Select u ∈ N \ X satisfying d(u) = min{d(v) | v ∈ N \ X}
7 |  X ← X ∪ {u}
8 |  for (u, v) ∈ E do
9 |  |  d(v) ← min{d(v), d(u) + w(u, v)}
```

It is shown that the final value of $d(u)$ is the length $\textsc{dist}(u)$ of a shortest path from s to u. In a sense, this algorithm is *node oriented*, since operations in each iteration of the while loop are determined based on the node $u \in N \setminus X$ found in Line 6, and all selected nodes are collected in the set X. Moreover, observe that each edge $(u, v) \in E$ is applied in Line 9 exactly once. Without changing the correctness, an *edge-oriented* version of this algorithm can be obtained by a simple modification.

Algorithm 2: An Edge-Oriented Version of Dijkstra's Algorithm

```
1 d(s) ← 0
2 for u ∈ N \ {s} do
3 |  d(u) ← ∞
4 Y ← ∅
5 while Y ≠ E do
6 |  Select (u, v) ∈ E \ Y satisfying d(u) = min{d(x) | (x, y) ∈ E \ Y}
7 |  Y ← Y ∪ {(u, v)}
8 |  d(v) ← min{d(v), d(u) + w(u, v)}
```

Observe that, at the end of each iteration of the while loop, each $d(u)$ is the length of a shortest path from s to u in the directed graph (N, Y). Let $\textsc{dist}_Y(u)$ be the length of a shortest path from s to $u \in N$ in (N, Y), and let

$$E_Y = \{(x, y) \in E \setminus Y \mid \textsc{dist}_Y(x) = \textsc{dist}_Y^{\min}\},$$

where $\textsc{dist}_Y^{\min} = \min\{\textsc{dist}_Y(x) \mid (x, y) \in E \setminus Y\}$. Then, Line 6 can be reformulated as follows.

- Select $(u, v) \in E_Y$.

According Line 8, we have

$$\textsc{dist}_{Y\cup\{(u,v)\}}(x) = \begin{cases} \min\{\textsc{dist}_Y(v), \textsc{dist}_Y(u) + w(u, v)\} & \text{if } x = v, \\ \textsc{dist}_Y(x) & \text{otherwise.} \end{cases}$$

Now let switch our focus to our branch and bound algorithm. Let (Y, σ) be a partial solution maintained in the algorithm, and let $\textsc{dist}_{(Y,\sigma)}(u)$ be the length of a shortest path from s to $u \in N$ in (N, Y) with all edges in $\sigma \subseteq Y$ are interdicted, i.e., the weight function w is defined as follows.

$$w(u, v) = \begin{cases} \alpha(u, v) + \beta(u, v) & \text{if } (u, v) \in \sigma, \\ \alpha(u, v) & \text{otherwise.} \end{cases}$$

By initializing (Y, σ) with $(\emptyset, \emptyset)$, we select an edge (u, v) from $E_{(Y,\sigma)}$ at each branching stage.

$$\begin{aligned} E_{(Y,\sigma)} &= \{(x, y) \in E \setminus Y \mid \textsc{dist}_{(Y,\sigma)}(x) = \textsc{dist}_{(Y,\sigma)}^{\min}\}, \\ \textsc{dist}_{(Y,\sigma)}^{\min} &= \min\{\textsc{dist}_{(Y,\sigma)}(x) \mid (x, y) \in E \setminus Y\}, \end{aligned}$$

and based on $\textsc{dist}_{(Y,\sigma)}(x)$s, $\textsc{dist}_{(Y\cup\{(u,v)\},\sigma)}(x)$s and $\textsc{dist}_{(Y\cup\{(u,v)\},\sigma\cup\{(u,v)\})}(x)$s can be maintained by updating at most two of them.

Obviously, the total interdiction cost $C(\sigma)$ can be maintained as well during the process. As a side effect, when (Y, σ) become a complete solution, i.e., $Y = E$, the length $L(\sigma) = \textsc{dist}_{(Y,\sigma)}(t)$ of a shortest path from s to t with all edges in σ are interdicted and the total interdiction cost $C(\sigma)$ are obtained as well.

3.2 Branch Elimination

Let (Y, σ) be a partial solution is obtained at some stage, and an edge $(u, v) \in E_{(Y,\sigma)}$ is selected for branching into $(Y \cup \{(u, v)\}, \sigma)$ and $(Y \cup \{(u, v)\}, \sigma \cup \{(u, v)\})$. Suppose $\mathrm{DIST}_{(Y,\sigma)}(v) \leq \mathrm{DIST}_{(Y,\sigma)}(u) + \alpha(u, v)$. Then, we have

$$\begin{aligned} \mathrm{DIST}_{(Y\cup\{(u,v)\},\sigma)}(v) &= \min\{\mathrm{DIST}_{(Y,\sigma)}(v), \mathrm{DIST}_{(Y,\sigma)}(u) + \alpha(u, v)\} \\ &= \mathrm{DIST}_{(Y,\sigma)}(v), \end{aligned}$$

and, from $\beta(u, v) \geq 0$, we have $\mathrm{DIST}_{(Y\cup\{(u,v)\},\sigma\cup\{(u,v)\})}(v) = \mathrm{DIST}_{(Y,\sigma)}(v)$ as well. It implies that $\mathrm{DIST}_{(Y\cup\{(u,v)\},\sigma)}(x) = \mathrm{DIST}_{(Y\cup\{(u,v)\},\sigma\cup\{(u,v)\})}(x)$ for each $x \in N$. Moreover, for all $\pi \subseteq E \setminus \{(u, v)\}$ satisfying $\sigma \subseteq \pi$, we have $L(\pi) = L(\pi \cup \{(u, v)\})$, but $C(\pi \cup \{(u, v)\}) = C(\pi) + c(\{u, v\}) > C(\pi)$, i.e., the interdiction strategy $\pi \cup \{(u, v)\}$ is not Pareto optimal since it is dominated by the interdiction strategy π. Therefore, the branch to $(Y \cup \{(u, v)\}, \sigma \cup \{(u, v)\})$ can be eliminated when $\mathrm{DIST}_{(Y,\sigma)}(v) \leq \mathrm{DIST}_{(Y,\sigma)}(u) + \alpha(u, v)$.

3.3 Our Proposed Algorithm

Algorithm 3: Our Proposed Algorithm

```
1  Function main()
2      d(s) ← 0
3      for u ∈ N \ {s} do
4          d(u) ← ∞
5      P ← ∅
6      BranchAndBound(∅, ∅, P, d, 0)
7      return P
8  Function BranchAndBound(Y, σ, P, d, C)
9      if Y = E then
10         P ← NonDominatedSet(P ∪ {σ})
11     else
12         Select (u, v) ∈ E_(Y,σ)
13         D ← d(v)
           // The first branch
14         d(v) ← min{D, d(u) + α(u, v)}
15         BranchAndBound(Y ∪ {(u, v)}, σ, d, C)
           // The second branch
16         if D ≠ d(v) then
17             d(v) ← min{D, d(u) + α(u, v) + β(u, v)}
18             BranchAndBound(Y ∪ {(u, v)}, σ ∪ {(u, v)}, d, C + c(u, v))
19             d(v) ← D
```

4 Results

We implemented this proposed approach in three networks that were also used in [10] (Figure 1). Table 1 shows the output of our proposed algorithm and existing algorithm by Rocco *et al.* [10] applied to Network 1 of Figure 1. In this output,

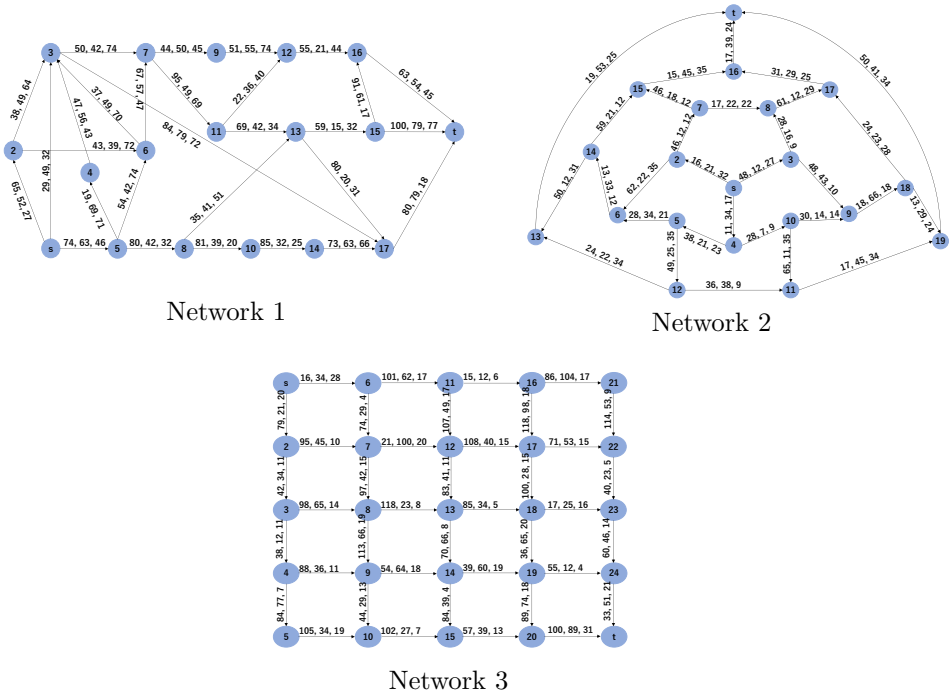

Figure 1: Networks for numerical experiments

the length after interdicting any edge is set to ∞ (the link is destroyed), and the existing algorithm's output is implemented in [10]. This exact algorithm provides the optimal Pareto set, which is high-quality than that of the existing algorithm. This section considers the computation time of this algorithm on networks with different total number of edges. Table 2 shows the calculation time when this algorithm was implemented on the three networks(Figure 1). In the paper, the calculation time for each network is not described, but the maximum CPU time to obtain a solution to each network is described as 20 s, so it can be said that the existing algorithm is implemented in 20 s or less. Note that our proposed exact algorithm can be implemented in three networks with a total number of edges of 40 or less, with computation time equal to or better than that of the existing algorithm. Our proposed algorithm is implemented in a Java based language, namely Processing, and by running it on a Macintosh with 2.9 GHz Intel Core i9,

the results shown in those tables are obtained.

distance	cost	Non-dominated Interdiction strategies	
		Our Approach	Rocco *et al.* [10]
193	0		
292	18	$\{(17, t)\}$	$\{(17, t)\}$
348	50	$\{(s, 3), (17, t)\}$	$\{(s, 3), (17, t)\}$
366	82	$\{(s, 3), (13, 15), (17, t)\}$ $\{(s, 3), (5, 8), (17, t)\}$	$\{(s, 3), (13, 15), (17, t)\}$
∞	94	$\{(12, 16), (13, 15), (17, t)\}$	
∞	95		$\{(13, 15), (16, t), (17, t)\}$

Table 1: Results for Network 1

network (total number of edges)	1(30)	2(30)	3(40)
calculation time(s)	0.05	0.66	58.87

Table 2: Calculation Time

5 Conclusions

In this paper, we proposed an exact algorithm for the bi-objective shortest-path network interdiction problem (maximization of shortest-path length and minimization of interdiction strategy cost). Based on the results, our proposed exact algorithm can provide the optimal Pareto set for the problem in reasonable time when the total number of edges is 40 or less. This solution allows network decision makers to fully understand the impact of each strategy on network performance.

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On Experimental Part of Behavior under Ambiguity

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Abstract

People are risk-takers, risk-averse, or neutral. In the literature, one can find experiments illustrating the ambiguity aversion of human decision-makers. Recently, a personal coefficient of ambiguity aversion has been introduced. We have decided to measure the coefficient and its stability during the time. In this paper, we describe performed experiments and their structure to launch a discussion of possible design weaknesses or to suggest other methods of measuring it.

1 Introduction

When designing new methods of artificial intelligence, it is necessary to know how people behave when they have to make decisions and do not have enough information to do so. In this case, the fact of whether the decision-maker is either risk-taker or risk-averse has a huge influence.

It is generally accepted that the basis for the normative decision-making theory was laid by [25], and (ten years later) [18] who has developed the idea that the behavior of a decision-maker can be modeled with the help of subjective probability and utility functions. Such a subjective setting explains why different decision-makers accept different decisions but all of them, if they are rational, use the same decision criterion: all of them maximize the *expected utility*. Since that time, many papers presenting situations with human decision-makers not following this principle, have been published. Let us mention here just some of these papers like [14, 1, 9, 4] (the last one introduces the famous Ellsberg's example/paradox, which is generally accepted as evidence for ambiguity aversion), and especially the papers introducing the prospect theory [15, 16] (Nobel prize 2002). The phenomenon of ambiguity aversion is connected with the fact that human decision-makers do not like ignorance. They usually prefer uncertainty connected with randomness than total ignorance.

We believe that the ambiguity aversion phenomenon [4, 5, 6, 14, 13] is closely connected to the fact that classical probability theory has difficulties with representing ignorance, or vagueness [19]. And it is this shortcoming, which causes why some decision theorists - using probability theory as the main theoretical tool - consider human decision-making behavior paradoxical. The importance of a "more powerful" tool for decision-making starts to be obvious to a wide range of users. E.g. [8] argued that traditional approaches to decision-making based on expected utility maximization are out of their depth in the area of environmental policy, *"as they force us to act as if we know things that we know we do not"*.

The goal of our current research is to find a way to create mathematical models proving the same ambiguity aversion as human decision-makers. The method of our research is described in detail in our other paper in these proceedings that contains the preliminary results from performed experiments [11]. This paper tries to describe the way how the experiments were performed and how they evolve during the time. It also brings a description of the testing tool - a web-based application.

2 Belief functions

The theory of belief functions [19, 3, 21] (and it does not matter at this moment whether we consider Dempster-Shafer theory of evidence, or if we understand belief function as a generalization of a probability theory based on the concept of credal sets) was designed to describe situations under vagueness and/or ignorance. Therefore, there is no surprise that situations described by Allais, or Ellsberg can be well represented in this theory. This is also the reason why we have decided to apply the theory of belief functions to model subjective human decision-making under ambiguity (as suggested already by Thomas [24])

The idea of representing the knowledge in the form of belief function is not new. Nevertheless, it is of great importance to have a tool how to compute the expected utility from it. To do so, we use the approach suggested in [12, 10]). We assume that the reader is familiar with at least the foundations of this approach. Therefore, we introduce just the notation used in this paper.

The theory of belief functions [19] can be interpreted as a generalization of probability theory [7], or within another nonadditive uncertainty theory having a possibility to represent situations that are connected with the terms like vagueness, ignorance, or ambiguity. The same role that is played by a probability distribution (measure) in probability theory can be played by several functions in the theory of belief functions. In our brief exposition, we will do just with three of them: basic probability assignment, belief and plausibility functions.

Suppose X is a random variable with state space Ω_X. Let 2^{Ω_X} denote the set of all *non-empty* subsets of Ω_X. A *basic probability assignment* (BPA) m for X is a function $m : 2^{\Omega_X} \to [0, 1]$ such that

$$\sum_{\mathsf{a}\in 2^{\Omega_X}} m(\mathsf{a}) = 1.$$

The subsets $\mathsf{a} \in 2^{\Omega_X}$ such that $m(\mathsf{a}) > 0$ are called *focal* elements of m. An important example of a BPA for X is the vacuous BPA for X, denoted by ι_X, such that $\iota_X(\Omega_X) = 1$. It corresponds to total ignorance. If all focal elements of m are singletons (one-element subsets) of Ω_X, then we say m is *Bayesian*. In this case, m is equivalent to a probability distribution.

In the theory of belief functions, the fact that $\mathbf{a} \subseteq \Omega$, for which $|\mathbf{a}| > 1$, is a focal element for BPA m, expresses our ignorance regarding how the probability mass $m(\mathbf{a})$ is distributed among the elements of set $\mathbf{a}$. For example, suppose $\Omega = \{x_1, x_2\}$, and BPA m is defined as follows: $m(\{x_1\}) = 0.2$, $m(\{x_2\}) = 0.3$, $m(\{x_1, x_2\}) = 0.5$. This m represents the knowledge that the probability of x_1 is at least 0.2 and at most 0.7, and the probability of x_2 is at least 0.3 and at most 0.8. We know nothing more, nothing less.

As said above, the information in a BPA m can be equivalently represented by corresponding *belief* and *plausibility* functions Bel_m and Pl_m, respectively that are defined as

$$Bel_m(\mathsf{a}) = \sum_{\mathsf{b}\in 2^{\Omega_X}\,:\,\mathsf{b}\subseteq\mathsf{a}} m(\mathsf{b}), \qquad Pl_m(\mathbf{a}) = \sum_{\mathbf{b}\in 2^{\Omega}:\,\mathbf{b}\cap\mathbf{a}\neq\emptyset} m(\mathbf{a}),$$

for all $\mathsf{a} \in 2^{\Omega_X}$. Notice that it is obvious that for all $\mathbf{a} \in 2^{\Omega}$, $Bel(\mathbf{a}) \leq Pl(\mathbf{a})$. If $Bel(\mathbf{a}) = Pl(\mathbf{a})$ then we are sure that the probability of $\mathbf{a}$ equals this value. Otherwise, the larger difference $Pl(\mathbf{a}) - Bel(\mathbf{a})$ the more ambiguity about the value of the probability of $\mathbf{a}$. This follows from the fact that like Bayesian BPA corresponds to a unique probability distribution, a non-Bayesian BPA m corresponds to the following convex set of probability distributions on Ω called a *credal set* of BPA m ($\mathcal{P}$ denote the set of all probability distributions on Ω):

$$\mathcal{P}(m) = \left\{ P \in \mathcal{P} : \sum_{x\in\mathsf{a}} P(x) \geq Bel_m(\mathsf{a}) \text{ for } \forall \mathsf{a} \in 2^{\Omega} \right\}.$$

3 Decision making

Savage's decision-making theory [18] is based on the computation of expected utility. To this end, we need a respective (subjective) probability distribution. In our approach, we follow the same basic idea, but to compute a value of decision criterion we do not use a subjective probability distribution but another function that, from the mathematical point of view, manifests properties of a *superadditive capacity*. Such a function is deduced from the credal set $\mathcal{P}(m)$ in two steps described below [13]. In the first step, we select a probability distribution, which in a way represents the knowledge from the considered credal set – this is done by a

probability transform – and by a subsequent subjective reduction that models an ambiguity aversion of a decision-maker.

In the belief function theory, there are several methods how to find a mapping that assigns a probability distribution to each BPA [2]. As examples, let us mention just two of them: well-known *Maximum entropy* element of $\mathcal{P}(m)$, and *pignistic transform*, advocated by [22, 23], defined by the formula

$$Bet\_P_m(x) = \sum_{\mathbf{a}\in 2^\Omega : x\in\mathbf{a}} \frac{m(\mathbf{a})}{|\mathbf{a}|}.$$

Notice that the importance of the latter transform is stressed by the fact that, as it was recently proved by [17], it coincides with the famous Shapley value [20] known from the game theory.

As already explained, the maximization of the expected utility does not correspond to the human way of decision-making. Therefore, we do not use a probability distribution to compute an expected value. We assume that the ambiguity aversion makes a decision-maker to underestimate the probabilities of some events. The greater ambiguity, the greater underestimation. Therefore we reduce a probability P_m (got by some of the above-introduced probability transforms) to get a personalized weights

$$r_{m,\alpha}(x) = (1-\alpha)P_m(x) + \alpha Bel_m(\{x\}),$$

where the coefficient $\alpha \in [0,1]$ reflects the level of the ambiguity aversion of a considered decision-maker. Notice, that the amount of reduction depends not only on the ambiguity aversion coefficient α but also on the amount of ignorance associated with the state x. If we are certain about the probability of state x, it means that $P_m(x) = Bel_m(\{x\})$, the corresponding probability is not reduced: $r_{m,\alpha}(x) = P_m(x)$. On the other hand, the maximum reduction is achieved for the states connected with maximal ambiguity, i.e., for the states for which $Bel_m(\{x\}) = 0$.

4 Experiments

People are different. Some are risk-takers, some are risk-averse. We believe that the behavior can be modeled using the theory of belief functions and that each decision-maker has different strength of ambiguity aversion – possibly expressible using coefficient α defined above. We even found some people with negative coefficient using our experiments - they are risk-takers. Using the following experiments we would like to prove or disapprove whether the ambiguity aversion is a personal characteristic of the decision-maker and whether it is possible to measure it.

As far as we know, nobody has studied an inter-temporal behavior of an individual decision-maker under different scenarios concerning ambiguity aversion up to now. Therefore, though keeping the anonymity of the experimental individuals,

we have designed our experiments to analyze the behavior of a decision-maker facing different problems, plus we plan to test one problem in different experimental sessions (e.g., half a year afterward). This was however not done so far. This should also testify whether the coefficient of ambiguity aversion (mentioned in the previous section) is a personal characteristic of a decision-maker.

What is important, no personal data are collected and stored. All persons participating the behavioral testing are identified by their identifiers (a sequence of characters of their personal choice). To be able to answer some statistical questions, the only information of personal character are age, sex, and education. The main information stored mirror the behavior of experimental persons in specified situations of a gamble.

4.1 Experiment design

Discussions with psychologists have revealed some interesting insights. People behave differently in real situations comparing to a presented hypothetical situation. How to achieve real behavior in a laboratory environment? One option is to use the concept of money by putting the participant's own money into the experiment. The fact of using their own money is crucial – the behavior when playing with artificial money or money belonging to someone else is different. Similarly, to minimize the influence of the ordering of the individual tasks, we have to give them to each participant in different random order.

Participation in the experiment is rewarded. People will receive 50CZK but this amount is paid before the experiment and it is emphasized that fact that this money belongs to them and they can keep it (and use it in the experiment as well). Using this we expect that the participants will feel that they play with your own money.

The experiment is realized in the form of a lottery. Imagine an urn with colored balls. Colors are known, information about the number of balls of each color varies according to the actual lottery. In total there are about 12 different scenarios. Players will receive all known information about the urn content and must decide which color to bet on and how much to bet. Depending on the amount of the bet, they participate in a real lottery. If the player guesses the color of the draw, he/she wins 100CZK.

The participants can use the money they received as a reward for participation as an input "capital" for the lottery games they are participating in. Of course, one can bet more than 50CZK in the sum to increase the change of winning. After betting, the lottery is played and any eventual winnings are paid out.

Because the participants receive game situations in a different order, the lotteries cannot be played immediately. The participants have to go through all the situations and bet on all games. Then the betting is closed and the lotteries are performed in reality. To do so, we have a real urn and real balls. Following the description of each situation, we randomly fill the urn and then one of the participants randomly drawn a ball. Note that this part is no longer important for our

needs and we do not collect information about winnings and losses. Nevertheless, it is vital for the real-life feeling of the participants.

4.2 Typical session

Typically, the session is organized as follows. After an introduction of the goal of the research, the participants are asked to select their identifier. The sessions are usually held in a computer laboratory so that each person answers the questions using a keyboard.

The assistant then gives out an information leaflet with the following text:

4.3 Information letter

Dear participants of ***Decision Making under Uncertainty*** *experiment.*

When designing new methods of artificial intelligence, it is necessary to know how people's behavior changes when they face varying degrees of lack of information. Therefore, we sincerely thank you for your help in participating in this experiment. Please, accept 50 CZK as a little reward for this help and also as an input "capital" for several lottery games on which the experiment is based. The assistant will reward you within the next few minutes.

Please note that this is a statistical ***anonymous survey****. We do not collect or store your personal information. Nevertheless, we would like to know if there is a difference in the behavior of men and women, students and mature managers. Therefore, we ask you for information about your sex, age, and education. Since we would like to know if you are always behaving the same (or alike), we would like to welcome you to participate in experiments repeatedly, so please also sign up with a nickname that you will remember for the next time.*

After you start your computer and sign in with your nickname, your computer presents you with a variety of situations. In each of them, you can participate in the draw and win 100 CZK.

How the given lotteries differ? For each lottery you will learn some (even incomplete) information about the contents of the lottery urn in which there are colored balls:

- *You will always find out whether balls that can appear in the urn are of three colors (black, white, yellow) or six colors (black, white, yellow, red, green, blue).*
- *You can learn how many balls are in the urn (but you don't have to).*
- *You can know the exact number of balls of one (or several) colors used in the draw (but you don't have to). In almost all situations, however, you will lack*

information on the ratio of other colors. In this case, one of the colors maybe not presented in the urn at all. For example, if you only know that there are eight balls in the urn of six possible colors, and just one of them is red, then maybe only balls of two colors are presented in the urn. Or you can imagine an urn with five balls only, each of six possible colors. Of course, one color may be missing at all.

For each lottery you must specify:

- *Which color is the winning color for you?*
- *How much you are willing to bet to participate in the game.*

When the betting is finished by all participants, all the draws will be realized (we cannot realize them during the data collection because the individual situations are presented to you in random order). Therefore, when you fill in the data on your computer, think well about each situation, because you have a chance to win, but also to lose real money. Only a part of you is involved in each draw. The bigger the amount you bet, the more chance you will participate in the draw. On the other hand, you also risk this amount if your color is not drawn. You are no longer allowed to withdraw from the game during the draw. If the computer selects you in the game, it will deduct the money you bet and, in case of a win, it will credit you 100 CZK. Be aware that lotteries are designed so that the vast majority of you have a big chance of winning (even over 300 CZK). However, some of you will lose (although losing more than the 50 CZK you received as an entry reward is unlikely - even though it has already happened). Any winnings and losses are settled with the assistant after the experiment.

After everyone has read the information letter, the web-based application is launched and participants go through different lotteries and answer questions about selected color and bet. Recall that each participant receives lotteries in random order to minimize the impact of the order on experiment results. In the following example, you can see a typical lottery from the experiment.

In the experiment, we want to estimate the aversion to uncertainty. To calculate it, we need to get the maximum amount the player is willing to bet on his chosen color. So, we want to push him into the highest bet he is still willing to make. We do this by limiting the number of participants in a real lottery. At the beginning of the experiment, it is announced that the number of participants in each real game will be limited to about a third of all participants, mainly based on the amount staked. More precisely, the greater part of the players is selected based on the amount staked (the larger the bet, the more likely you are to play a real game). The smaller part is then selected randomly.

So if you don't bet enough money in the game, it's highly likely that you won't participate in real games at all. So you can neither to win nor to lose.

4.4 Example

Figure 1 illustrates the design of the application. Because we expect to have Czech participants only, the application is in Czech only so far. The description of the lottery can be translated as follows:

Experiment

Task 1/8 - (situation no. 9)
Urn A
Number of balls:

- *total number:* ***9***
- *number of colors:* ***6***
- *red balls:* ***exactly one***
- *numbers of balls of other colors are unknown*

Question
Choose a color. If the randomly drawn ball has the color you have selected, you win 100 CZK. How much are you maximally willing to pay to take part in the lottery?

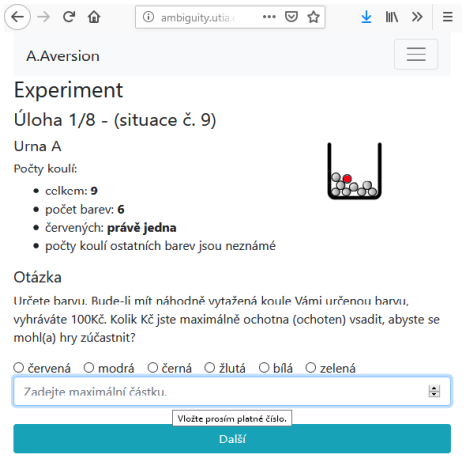

Figure 1: Screenshot from the application

Then you can see the list of six colors and a text input form for your bet. You submit your bet and color by clicking on the button. Then the next task is shown. The lottery is illustrated by a sketch of an urn with 9 balls. Exactly one of them is red and the rest is gray to emphasize that we have no information about their colors.

5 Experiences

During the first few experiments, we have discovered several problems in the experiment settings.

- The game labeled as **Rn**, which contains several lotteries, was split into separate lotteries in the first experiment. The results of the experiment were confusing and inconsistent across participants. It seems that people are not able to keep in mind the individual variants of this game and therefore the associated game Rn was created. The variants are next to each other.

- In the second experiment, we did not sufficiently emphasize the fact that only a part of the participants with the highest bet will participate in the draw. This led to the printed manual you saw in the previous section.

ID	Ignorance Red	Ignorance Color	Uniform 30 Red	Uniform 30 Color	One red in 5	One red in 6	One red in 7	One red in 8	One red in 9	One red in 10	One red in 11	One red in 12	Ellsberg One color	Ellsberg Two colors
1	10	10	10	10	5	5	5	5	5	5	5	5	15	15
2	5	7	16	10	1	1	1	1	1	1	1	1	8	17
3	3	1	16	16	20	16	13	12	11	10	9	8	33	66
4	0	0	20	20	10	0	0	0	0	0	0	0	10	30
5	1	1	1	20	1	1	1	1	1	1	1	1	1	10
6	10	10	16	16	19	15	14	12	10	10	10	10	32	20
7	16	15	16	11	6	10	6	2	5	3	2	7	15	11
8	10	20	10	12	15	8	4	5	5	5	5	5	20	20
9	3	6	20	17	10	5	5	4	3	2	1	1	23	15
10	3	3	16	15	19	15	14	12	11	9	9	8	33	33
11	1	0	5	5	5	10	0	0	0	0	0	0	5	5
12	6	2	0	9	2	6	1	1	1	1	1	1	10	13
13	14	14	14	14	19	14	14	14	15	15	15	15	24	24
14	1	3	19	2	16	8	11	8	7	8	9	5	26	7

Figure 2: An overview of bets on selected colors in one experiment.

- In another experiment, we came across the fact that some of the participants were playing some sort of alternative game, trying to guess how much the others would bet and bet accordingly.

- During standard experiments with students, the average profit was around CZK 130 per person. An interesting situation occurred in an experiment carried out during the traditional seminar of our institute. In this case, perhaps no one left the experiment with some winnings. Some participants lost more than 150 CZK. We are not sure about the reasons for this to happen, however, education does not seem to be an asset. Thanks to the knowledge of probability theory, the participants bet amounts corresponding to the probability of drawing the color. Unfortunately, this is not a good approach even in the long run, because you win as much as you loose only. Moreover, the draw is done only once in our case.

  Above that, even in the case of Ellsberg's case of variant E2 with a chance to win of $\frac{2}{3}$ in the case of choosing black color, the red color was finally drawn.

6 Experiment application

As already mentioned above, we created a web-based application for the betting. You can find it at http://ambiguity.utia.cas.cz/. It is a simple application written in PHP scripting language that is especially suited to web development. Data are stored in MySQL database that has 6 tables. The structure of the database is illustrated by Figure 3.

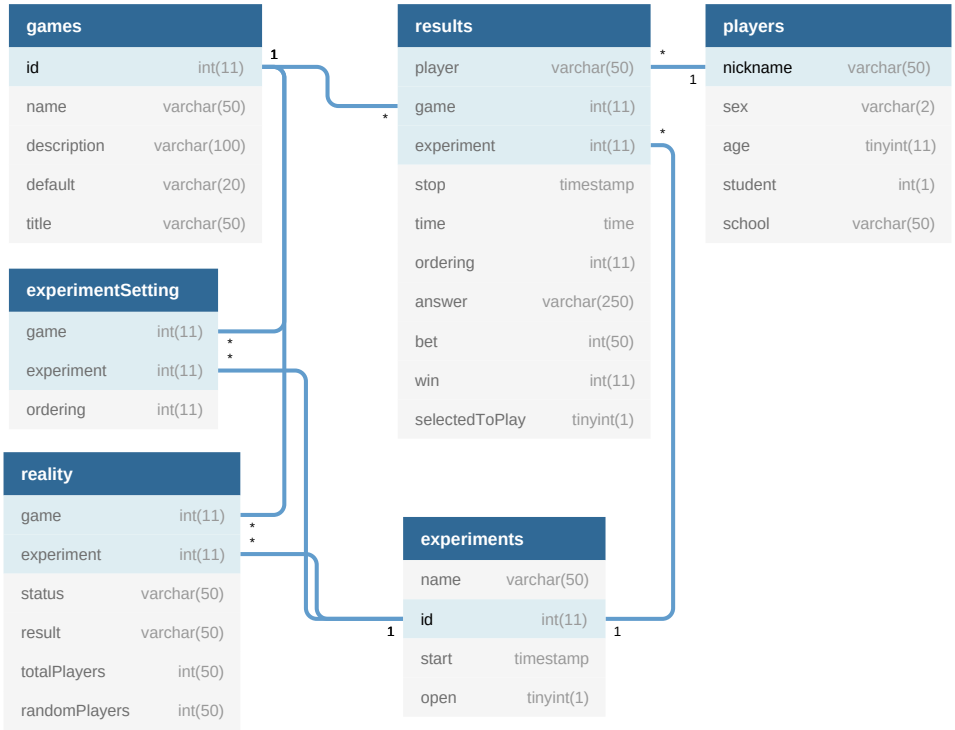

Figure 3: Database structure

- `players` - the list of participants. The table has 5 columns: nickname, sex, age, student (Yes/No), school

- `games` - the list of possible situations in the system. The table has 5 columns including unique integer identifier, name, and description

- `experiments` - the list of experiments. The table has 4 columns including unique integer identifier, name, date, and a flag whether the experiment is running (Yes/No)

- `lotterySetting` - it specifies the games(lotteries) assigned to each experiment

- `results` - table with bets and answers. It has 10 columns: game, player, time needed to finish the task, in which ordering the question arrived, etc.

7 Conclusion

Work on experiments is still ongoing. However, it is already clear that it is possible to measure a personal coefficient expressing something like ambiguity aversion and the answers seem to be consistent with this coefficient. In the future, we plan to conduct further experiments with the same participants to see where the coefficient changes over time. We are also planning to add more variants and other situations containing ambiguities.

Acknowledgement

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Inconsistency Distribution in Saaty's Pairwise Comparison Matrices

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Abstract

Pairwise comparisons method is widely used to obtain ratings of alternatives or criteria in multiple criteria decision making. The quality and reliability of human experts' opinions are measured by (in)consistency indices. Saaty suggested eigenvector and eigenvalue of the pairwise comparison matrix for both priority vector and inconsistency computation — he introduced the Consistency Index and the Consistency Ratio and suggested CR = 0.1 as a maximal acceptable level of inconsistency in his Analytic Hierarchy Process. It is shown, that consistent ratings in the pairwise comparison matrix are localised in specific regions of the rating space, depending on the matrix dimension and the degree of the preference rating. Moreover, the relative frequency of acceptable consistent ratings significantly decreases as the rating intensity increases.

1 Introduction

Decision making is one of the most common activities of all living beings. To support human rational decision making, many formal methods have been proposed so far. Pairwise comparisons method represents a relatively simple, formally acceptable and widely applied approach for expert based rating of alternatives or criteria in multiple criteria decision models (MCDM). From the very beginning, much attention has been paid to computational algorithms for the transformation of the pairwise comparisons set to the "optimal" priority vector. Concurrently, the quality and reliability of human experts' opinions were studied and measured by many different (in)consistency indices [3]. Moreover, several methods for minimizing the inconsistency of human subjective ratings have been suggested [13]. Saaty [10], for example, prioritized eigenvector and eigenvalue approach for both priority vector and matrix inconsistency computation -– he introduced the Consistency Index (CI) and the Consistency Ratio (CR) and suggested CR = 0.1 as a maximal acceptable level of inconsistency in his Analytic Hierarchy Process (AHP).

The research gap can be identified in the previous works where most authors focused their attention to bring various methods and algorithms with the intention of minimizing the existing inconsistency of the pairwise comparison matrix (either obtained from randomly generated matrices by simulations [12] or from respondents in real-world experiments [4] or as a combination of the both [6]).

Many review papers have dealt with inconsistency indices for pairwise comparison matrices [2], [3], [1] and their axiomatization [7] so far. However, as far as is known, no systematic analysis of inconsistency distribution in pairwise comparison matrices has been done yet.

2 Methodology

For our analysis, MATLAB® environment and its corresponding scripting language were utilized to model pairwise comparison matrices and compute their inconsistency values for all possible evaluations. Some post-processing and visualizations of generated data were also provided in the EXCEL 2016 spreadsheet environment.

From many available consistency measures ([9], [2], [5]) we have selected the original Saaty's approach for this study as it is broadly accepted and has been successfully implemented in many real-world scenarios [11].

3 Definitions and preliminaries

In the following, we assume that $A = (a_{ij})$ is a positive multiplicative pairwise comparison matrix (PCM) of size n. Further, Saaty's fundamental scale [8] with integer values $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (where 1 stands for "A_i is equal to A_j" and 9 stands for "A_i extremely exceeds A_j") for pairwise comparisons is considered and the eigenvalue approach to inconsistency computation is used.

The **Consistency Index** (CI) was defined by Saaty [10] as

$$\mathrm{CI}(A) = \frac{\lambda_{max} - n}{n - 1} \tag{1}$$

where λ_{max} is the principal eigenvalue of A; $Aw = \lambda_{max} w$; $\mathrm{CI} \geq 0$.

For completely consistent rating the following relation must hold:

$$a_{ij}.a_{jk} = a_{ik}, \forall i, j, k \tag{2}$$

and then $\lambda_{max} = n$ and $\mathrm{CI} = 0$.

The **Consistency Ratio** (CR) is a standardized version of CI. We obtain CR by dividing CI by a real number RI (Random Index) where RI is calculated as an average CI of a very large number of randomly generated reciprocal matrices of the same size n:

$$\mathrm{CR}(A) = \frac{\mathrm{CI}(A)}{\mathrm{RI}} \tag{3}$$

Table 1: Values of RI for different number of random PCMs used in the computation. Adopted from selected authors [1] compared to own results.

n	Saaty (500)	Alonso, Lamata (100,000)	Mls (1,000,000)
3	0.58	0.5245	0.52388
4	0.90	0.8815	0.88332
5	1.12	1.1086	1.10816
6	1.24	1.2479	1.24848
7	1.32	1.3417	1.34045
8	1.41	1.4056	1.40417
9	1.45	1.4499	1.45059
10	1.49	1.4854	1.48587

The **Random Index** has already been experimentally generated by several authors [1] with varying results depending on the computational method and on the number of generated random matrices involved in the process. For inconsistent matrices, the number of acceptable ratings depends not only on the proposed limit value of CR (e.g. 0.1) but to a certain extent also on the computed value of RI. Therefore, before analyzing inconsistencies of reciprocal PCMs, values of Random Indexes for 1,000,000 random matrices were computed and compared with the previously published results. It can be seen (Table 1) that the differences in RI between 100,000 and 1,000,000 random matrices are very small (less than 0.1% in most cases). Nevertheless, even such small deviation may result in about 5% increase of the number of acceptable ratings. Therefore we used our RI values rounded to four decimal places in all subsequent simulations.

Assuming that the pairwise matrices A are reciprocal, only $m = \frac{n^2-n}{2}$ of their elements a_{ij} has to be evaluated. Moreover, each element, evaluated by Saaty's fundamental scale can theoretically reach any of 17 different numerical values: $a_{ij} \in \{1/9, 1/8, 1/7, 1/6, 1/5, 1/4, 1/3, 1/2, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. It leads to 17^m different ratings of the pairwise matrix A. For example, considering the matrix A, $dim(A) = 3$ we have $17^3 = 4913$ possible rating combinations of which only 85 (1.73%) are completely consistent (CI = CR = 0). For practical reasons, some small tolerance for the rating inconsistency was recommended by Saaty. We say, that all ratings with CR ≤ 0.1 are sufficiently consistent. In the case $dim(A) = 3$, the number of acceptable (sufficiently consistent) ratings then rise to 1021 (20.78%).

As the matrix size n is increasing, the number of possible rating combinations (rating space) rise exponentially and from $n \geq 5$ the complete (in)consistency analysis becomes practically impossible (Table 2). Therefore, just 3D rating space inconsistency distribution of the PCMs of size 3 will be analysed in more detail in this contribution.

Table 2: Number of all possible ratings of the reciprocal pairwise $n \times n$ matrix A for $n = 3, 4, ...10$.

n	No. of matrix elements for rating	No. of ratings
3	3	4913
4	6	241375569
5	10	$2.01599 \times 10^{+12}$
6	15	$2.86242 \times 10^{+18}$
7	21	$6.90919 \times 10^{+25}$
8	28	$2.83511 \times 10^{+34}$
9	36	$1.97770 \times 10^{+44}$
10	45	$2.34532 \times 10^{+55}$

4 Results

In the Matlab environment, .m functions were written:

- to generate 1,000,000 random reciprocal matrices and to compute average RI for matrix size $n = 3, ..10$ (Table 1),
- to generate all possible ratings of the PCM of size 3, to compute CI and RI and to show frequencies of consistent ratings as a function of rating intensity in one dimension (Table 3),
- to generate all possible ratings of the PCM of size 4, to compute CI and RI and to sum all consistent ratings.

Some generated data were also processed in MS Excel spreadsheet to make fast ad-hoc analyses and visualizations. In the Table 2, dimensions of the considered rating spaces are quantified. In the Table 4, absolute and relative numbers of acceptable consistent ratings in rating sub-spaces, computed from the center of the rating space are arranged. Finally, Figure 1 displays consistent ratings in two limit evaluations of the PCM.

5 Discussion and future work

The analysis of inconsistency distribution revealed some new knowledge. First, sufficiently consistent ratings (CR ≤ 0.1) in a PCM of size 3 are more (about twice) frequent in case that pairwise comparisons indicate weak or moderate intensity compared to very strong or extreme importance. Second, despite there is an increase in the absolute number of sufficiently consistent ratings (from 1021 in PCMs of size 3 to 760,913 in a PCM of size 4), relatively it corresponds to a decline from 20.78% to 3.15%.

Table 3: Frequencies of consistent ratings of the pairwise reciprocal 3×3 matrix A (289 ratings for each row).

$C1/C2$ Preference rating	No. of consistent ratings (CR$= 0$)	No. of consistent ratings (CR≤ 0.1)
1/9 and 9	3	45
1/8 and 8	4	45
1/7 and 7	2	49
1/6 and 6	4	46
1/5 and 5	2	52
1/4 and 4	5	56
1/3 and 3	6	69
1/2 and 2	8	94
1/1	17	109

Table 4: Distribution of consistent ratings (CR≤ 0.1) in 3D rating sub-spaces.

Distance d from the center	Total ratings $((2d-1)^3)$	Consistent ratings	Consistent ratings [%]
1	1	1	100.00
2	27	19	70.37
3	125	55	44.00
4	343	121	35.28
5	729	235	32.24
6	1,331	373	28.02
7	2,197	553	25.17
8	3,375	769	22.79
9	4,913	1,021	20.78

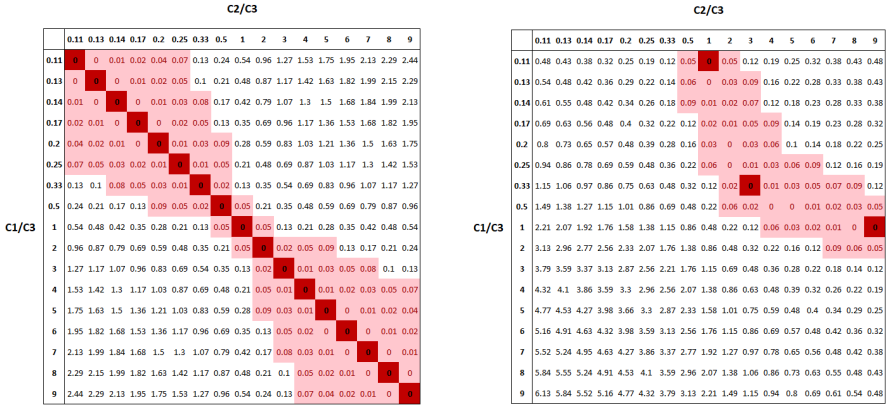

Figure 1: Consistency mapping of 3D rating space for $C1/C2 = 1$ (left) and $C1/C2 = 9$ (right).

Subsequent research in the field of (in)consistency distribution in pairwise comparison matrices will be focused on two particular and closely related problems - on the one hand, a new algorithm for identifying consistent ratings in larger matrices (with corresponding 10, 15, and more dimension rating spaces) will be searched. On the other hand, human expert evaluations of pairwise comparisons will be analysed from the rating intensity point of view with the intention to obtain new balancing coefficient for consistency index in the AHP.

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Proposal of probability risk Evaluation for System Development Project Based on Requirements Analysis and Bayesian estimation

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Abstract

For building a business system quickly, more efficient project management is needed. However, uncertain and subjective factors in requirements from customer cause cost overruns or schedule delays in the project. Furthermore, uncertain and subjective factors can lead to misunderstandings and false estimates when converting requirements into specifications or scheduling within a project. Thus this paper proposes a probabilistic risk evaluation method with Requirements Analysis and Bayesian estimation for project management to accurately evaluate the project risk. If this method works well, efficient project management will be realized.

1 Introduction

Business System development projects are challenging, because there are many requirements demanded from customers even while these requirements are proposed with the same priority. Additionally, requirements have subjective factors. Thus it is important to correct accuracy and prioritize requirements according to their essentiality and criticality to finish by schedule under budget. Although system developers estimate according to the complexity of projects[1][2], on the other hand customers expect the cost to be based on the number of requirements they demand. As a result, there is a difference between the customer and the system developer in the evaluation of estimate. Over cost or schedule delay is caused by missing estimate. Thus, evaluating risks based on requirement analysis properly is important to finish project successfully. Previous paper proposed cost share rate for projects based on requirements analysis in order to estimate and evaluate requirements accurately [3]. But the attention must be paid to the fact that estima-

tion has subjective factors, for example productivity and skills of the programmer. This paper proposes the cost share rate and Bayesian estimation to predict risks of project. Cost share rate is defined as the percentage of total cost assigned to each requirement. This paper shows the potential for evaluating risks of project.

2 Cost Prediction Methods for Projects

2.1 Previous research of project management

This paper demonstrates the potential to evaluate risk of a project using Bayesian estimation based on requirement analysis in system development project management. Previous research of project management focused by schedule, cost estimation or productivity [1]-[4]. Improving productivity contribute to finish project successfully [1]. There was also a previous research that deals with changing requirement in mechanical engineering design. This research showed one example of requirements analysis by exploring the possibility of predicting requirements change with graphical models of the requirement documents and historical change trends[4]-[8]. There is no research that refers uncertain and subjective factors in estimating based on requirement analysis. Therefore this paper proposes the method taking uncertain and subjective factors into account by the means of Bayesian estimation based on requirement analysis.

2.2 Typical Cost Prediction Methods

Usually the amount of program source code is predicted by some prediction method in order to assess projects [8]-[10]. Then the amount of program source code is converted into basic monetary cost. Next total cost is made by adding contingency budget to basic monetary cost. It is set as a budget for a project at first. In case of COCOMO method [8]. The amount of program source code acquired by this way would be converted into monetary cost using a parameter (PM: Person-Months) [8]. On the other hand, in case of Function Point method, the point are accumulated according to the complexity of system, for example the number of DB tables, dialog boxes, print forms and interfaces. Next, the acquired points would be converted into monetary cost using a parameter (PM: Person-Months) as those of COCOMO methods [8]. These method estimate according to complexity of systems, not according to their requirements.

3 Requirement Analysis and Evaluation with Cost Share Rate

3.1 Cost Share Rate

Cost overrun or schedule delay is caused by lots of changes in requirements. In particular, changing and uncertainty in essential requirement could bring a high risk. It is vital to distinguish the essential requirement which gives significant impacts to system specification or budget of the project. Thus, this research proposes the method to distinguish influential requirements that has high risks for cost share rate[3] Cost share rates indicate the impact of requirements based on requirements analysis. Costs for projects are usually estimated by grouping costs with the number of dialog boxes, interfaces or print forms and complexity based on requirements. Alternatively, costs are estimated by associating amount of cost with logic design, development, test, adjustment and documents of requirements, not according to the essentiality of requirements. System developers estimate according to the complexity of specifications, but customers expect the cost according to the number of requirements they demand. Thus, customers could not understand the estimates provided by system developers. This paper shows a method to calculate cost share rate for each requirement in order to evaluate requirements accurately for mutual understanding of the developer and customers. Cost share rate also indicates the importance and distinguishes essential requirements to prioritize properly. Requirements that have a high cost share rate must have a high risk, and should be also under strict control, because change or modification for essential requirement give impact to costs or the schedule.

3.2 Prediction of probability of the schedule delay for each task

In quantitative evaluation for risk management, risks are evaluated by the monitory loss[9] [10]. The monitory loss is defined as the damage multiplied by the probability of risk. Thus this section explains the trial for calculating risks by multiplying probability and cost share rate of each requirements. In this research, risk is considered as the probability of the schedule delay. Thus this research suppose that the probability of the schedule delay follows *beta* distribution (see Equation 1). In Figure 2 horizontal axis(x) shows the normalized start date of tasks. The vertical axis(y) shows the probability for the schedule delay against planed days for each task. Each task is categorized into process category. From these results the average of the probability for the schedule delay is considered to be 0.225. Thus this probability of 0.225 is assigned to standard requirements. Standard requirements are the requirements that belong to the design or programming, because they have average complexity and average risks for cost overrun and schedule delay.

$$f(x) = c \times x^{\alpha\ -1}(1-x)^{\beta\ -1} \tag{1}$$

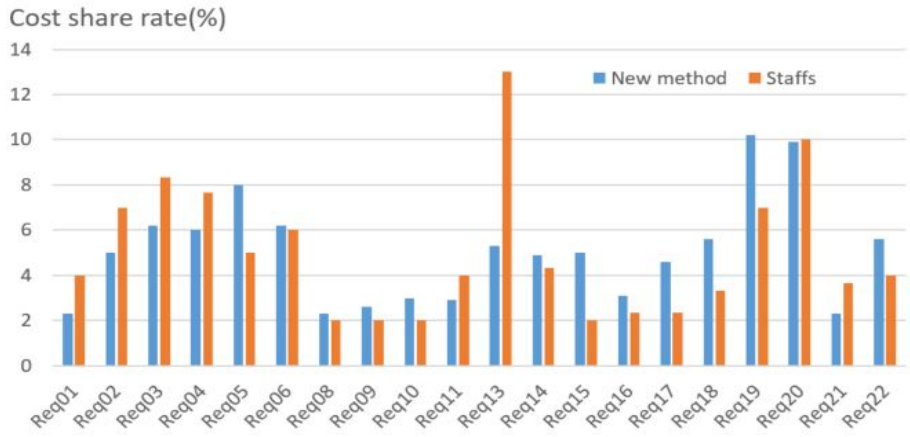


Figure 1: Cost share rate

Project risk is the defined possibility of cost overrun after this. On the other hand, In Table 2 requirement1 is about investigation and the main design. It indicates the probability of cost overrun risk as 0.1, because the uncertainty of the main requirement is the main reason for arranging an extra budget called contingency budget as 10% of total cost in addition to the basic budget. These past projects needed extra cost of 10% of total cost, because there are uncertainties at the beginning of the project. Because tasks are planned on the basis of requirements, it is important to evaluate requirements accurately in order to avoid the cost overrun. If requirements would be accurate, tasks would be set up appropriately, as a result the project could be finished successfully. Therefore, in this research, in order to avoid the cost overrun, it propose a method to categorize appropriately and evaluate requirements according to the degree of risk. Requirements should be categorized as the main design, design, development, and others according to their risk by the linguistic analysis and regression analysis.

Table 1: Obtained Parameters for β distribution

Coefficient	C	α	β
Value	0.840	1.021	3.048

3.3 Applying probability for another project

This section shows a trial to apply the probability of cost overrun to another small project which is building document management system. In this project, there are

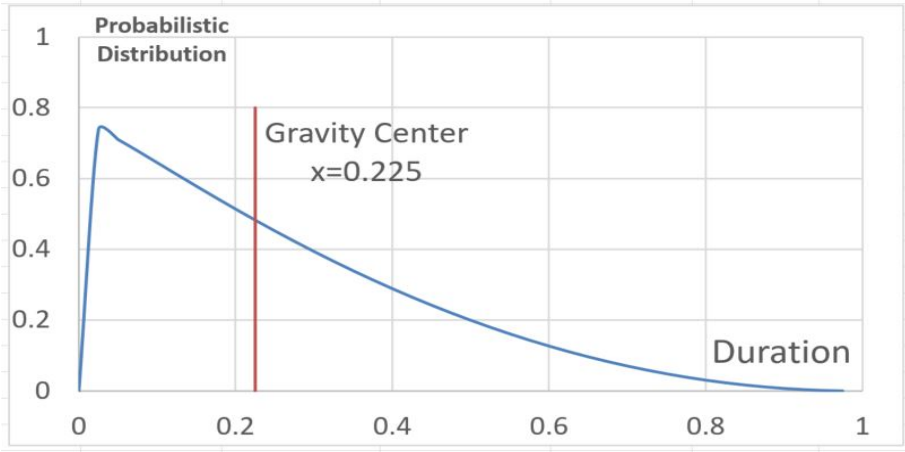


Figure 2: Curve for possibility of schedule delay

requirements2, requirements4, requirement7 and requirement8 which concern the design. These four requirements are standard requirements. Standard requirement is the requirement which has normal complexity and risks for the cost overrun. Thus, it has been assigned the probability of the schedule delay from β distribution as 0.225. The requirement 1 is concerning the main design and it has been assigned the probability of 0.1, because usually the schedule delay is caused by the uncertainty of the main design. Table 2 shows the estimated cost adapting the above mentioned probability of the schedule delay to each requirement. At the result, estimated total cost that includes expected monetary risk is about 110 ; on the other hand, actual total cost at completion is about 127(see Table 2). Thus from this result more efficient method is needed to estimate risk accurately.

4 Risks Evaluation of Projects using Cost Share Rate and Bayesian Estimation

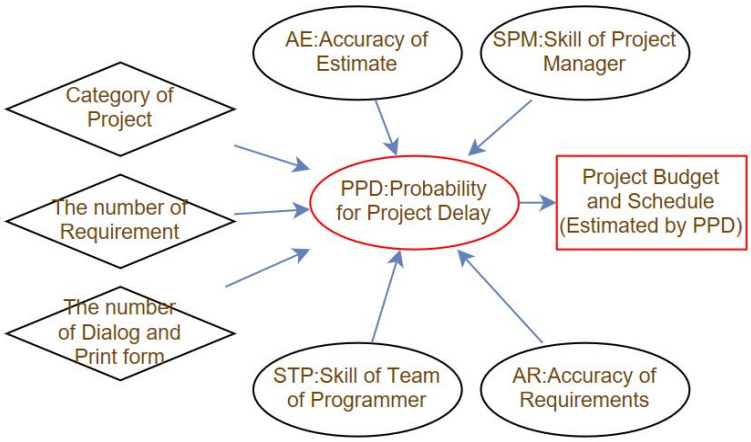
4.1 Influence diagram for system development project

This research considers an influence diagram that shows the relationship between factors in a project as shown in Figure 5 [11] . This influence model includes type of chance nodes as follows: AE (Accuracy of Estimate), SPM (Skill of Project Manager), STP (Skill of Team of Programmers), AR (Accuracy of Requirements). Additionally, this influence model includes decided nodes as follows: Category of Project, The number of Requirements, The number of Dialog and Print form, and decision node as Project Budget and Schedule (see Figure 3). In addition, this is

Table 2: Estimated cost using *beta* distribution

Require-ment No	Category	Cost share Rate(%)	Probability for Schedule Delay	Expected Monetary Risk	Estimated Cost	Actual cost
1	Main	6.4	0.1	0.6	7.0	10.6
2	Design	10.6	0.2	2.4	13.0	10.6
3	Program	2.1	0.0	0.0	2.1	2.1
4	Design	4.3	0.2	1.0	5.2	8.5
5	Program	2.1	0.0	0.0	2.1	2.1
6	Program	12.8	0.0	0.0	12.8	17.0
7	Design	6.4	0.2	1.4	7.8	8.5
8	Design	21.3	0.2	4.8	26.1	12.8
9	Program	12.8	0.0	0.0	12.8	25.5
10	Other	12.8	0.0	0.0	12.8	17.0
11	Program*	8.5	0.0	0.0	8.5	12.8
		100.0		10.2	110.2	127.7

*Program: Programming



Chance nodes(), Decision nodes(), Decided nodes().

Figure 3: Expanded Influence Diagram for projects

expanded, by adding Decided nodes(). Decided nodes has only information about projects, Decided nodes give no influence for calculating risks of projects.

4.2 Risk Evaluation with Bayesian network

Figure 4 shows the Bayesian network using Weka for the influence diagram shown in Figure 4 before given evidence, and it shows the conditional probability given by 3 [12] [13]. Numbers in Table 3 are obtained from two project managers subjectively. Thus Figure 5 called case1 shows the conditional probability for project delay is 0.13 when engineers skill is high and accuracy of requirement is standard. On the other hand, Figure 6 called case 2 shows the conditional probability for project delay is 0.29 when engineers skill is high and accuracy of requirement is low. For predicting risks of the project using Bayesian estimation and cost share rate, in this method, the expected monetary value for risk is obtained by cost share rate multiplied by the probability of schedule delay of whole project using Bayesian estimation with Weka. Table 4 shows the result of expected monetary value for risk is 0.13 when engineers skill is high with standard accuracy of the requirements (Case1). And the expected monetary value for risk is 0.29 when engineers skill is high with low accuracy of the requirements (Case2). This result, estimated cost is 129 is fit to seance of project manager. For example, actual cost is 127 (normalized) in Table 2 showed the case when engineers skill is high with low accuracy of the requirements. Although Figure 7shows the conditional probability for project delay is 0.07 when engineers skill is high, skill of project manager is high and accuracy of requirements is low(Case3). Also it shows that skill of project manager

is important for project.

Table 3: Conditional Probability from staffs

Probability Distribution Table for AE

	AE:Accuracy of Estimate			
Project Status	Low	Standard	High	Sum
Delay	0.6	0.25	0.15	1.0
On Schedule	0.1	0.2	0.7	1.0

Probability Distribution Table for SPM

	SPM:Skill of Project Manager			
Project Status	Low	Standard	High	Sum
Delay	0.65	0.25	0.1	1.0
On Schedule	0.1	0.3	0.6	1.0

Probability Distribution Table for STP

	STP:Skill of Team of Programmer			
Project Status	Low	Standard	High	Sum
Delay	0.6	0.3	0.1	1.0
On Schedule	0.1	0.3	0.6	1.0

Probability Distribution Table for AR

	AR:Accuracy of Requirements			
Project Status	Low	Standard	High	Sum
Delay	0.55	0.3	0.15	1.0
On Schedule	0.2	0.3	0.5	1.0

5 Conclusion

More efficient project management is needed in order to meet budget, finish by schedule, and maintain high quality in projects. Though it is difficult to evaluate risks accuracy, because there are uncertain and subjective factors in projects, especially requirements have much uncertainty. Thus this paper propose the cost share rates in order to distinguish essentiality of each requirements. In section 3, it

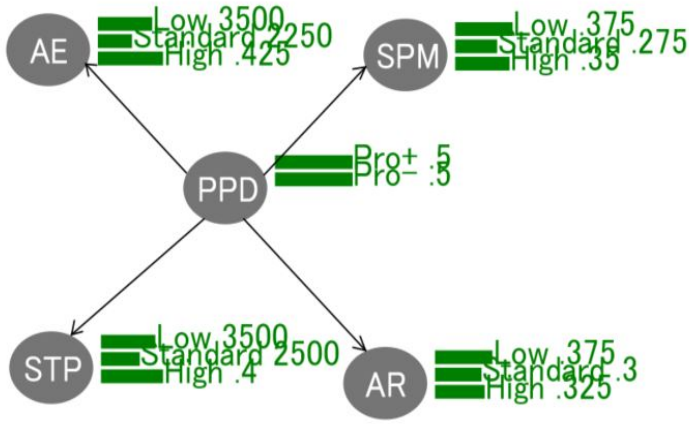


Figure 4: Bayesian network for the project(before given evidence)

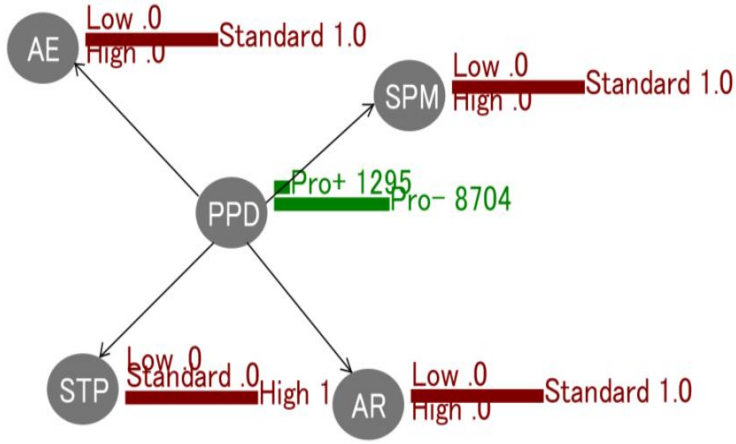


Figure 5: Bayesian network for the project (Case1)

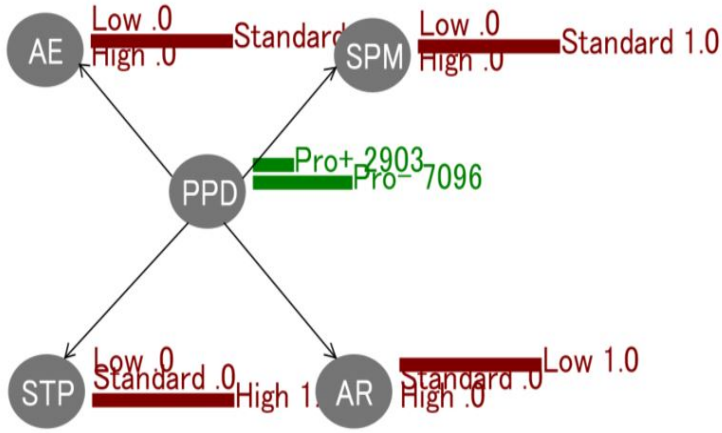


Figure 6: Bayesian network for the project (Case2)

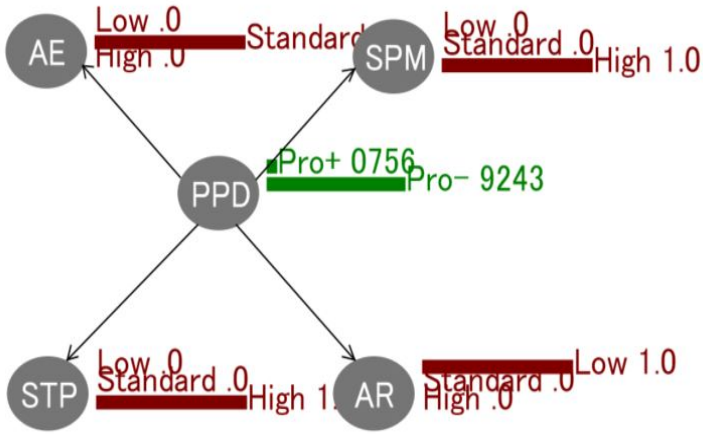


Figure 7: Bayesian network for the project (Case3)

Table 4: Probability of schedule delay using Bayesian estimation

NO	Requirement No	Category	A:Cost Share Rate(%) (from Table 5)	B:Probability for Schedule Delay (Case1)	Estimated Cost A*(1+B) (Case1)	C:Probability for Schedule Delay (Case2)	Estimated Cost A*(1+C) (Case2)	Actual cost
1	R1	Main	6.4	0.130	7.21	0.29	8.23	10.6
2	R2	Design	10.6	0.130	12.02	0.29	13.72	10.6
3	R3	Program	2.1	0.130	2.40	0.29	2.74	2.1
4	R4	Design	4.3	0.130	4.81	0.29	5.49	8.5
5	R5	Program	2.1	0.130	2.40	0.29	2.74	2.1
6	R6	Program	12.8	0.130	14.43	0.29	16.47	17.0
7	R7	Design	6.4	0.130	7.21	0.29	8.23	8.5
8	R8	Design	21.3	0.130	24.04	0.29	27.45	12.8
9	R9	Program	12.8	0.130	14.43	0.29	16.47	25.5
10	R10	Other	12.8	0.130	14.43	0.29	16.47	17.0
11	R11	Program	8.5	0.130	9.62	0.29	10.98	12.77
			100.00	1.43	113.00	3.19	129.00	127.7

shows it is difficult to predict risk of projects with cost share rate and probability for schedule delay is obtained from β distribution. Next this paper shows a trial to predict risk of projects accuracy using the cost share rate and Bayesian estimation. As a result, this paper shows the potential to evaluate the risk of projects accuracy using Bayesian estimation and cost share rate, in addition this method provides the opportunity to break down the cause of risks into risk factors [8] [9][10][14]. The conditional probability shown Table 3 includes subjective factor since it was obtained by a questionnaire filled in by project engineers, thus obtaining the conditional probability more logically is needed in future works. In addition, in order to manage projects better, further research is needed on risk evaluation with requirements analysis that takes into account uncertain factors and subjective factors in requirements.

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On Modelling of Syllogisms with "A few" and "Several"

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Abstract

In real world, interpretation of natural data using a natural language is very popular. In our previous papers, we introduced mathematical definitions of intermediate quantifiers, which were used for an analysis of natural data using fuzzy association rules. The main objective of this paper is to analyze new kind of generalized syllogisms with new intermediate quantifiers "A few" and "Several".

1 Introduction

The theory of a syllogism was, from the classical syllogisms point of view, already analyzed in the Aristotle's period, when it was Aristotle who proved the truthfulness of 24 basic syllogisms [17]. Later in the 20th century, several authors followed up this work and extended these basic syllogisms with new quantifiers and their related new shapes of true syllogisms. Most credit for this extension goes to Thompson and later Peterson [19, 16], who proposed 69 new syllogisms with intermediate quantifiers. Among these syllogisms, Thompson did not consider a possibility of two intermediate quantifiers in both premises. This work was later extended by Peterson in [16], where he proved another special group of so-called non-trivial syllogisms containing intermediate quantifiers in both premises. Verification of the truth of these syllogisms was constructed using Venn diagrams.

The theory of syllogisms was later extended for generalized quantifiers by several authors. The extensions were proposed in several directions. The first extension is generalization of the syllogistic reasoning in the four classical figures by replacing classical quantifiers by the generalized (fuzzy) ones. It was done in 1985 by L. Zadeh, who semantically analyzed a special class of syllogisms with intermediate

quantifiers in both premises ([22]), which we dealt with from the syntactical point of view in [9]. Zadeh's works were later elaborated by many authors. The extended syllogistic reasoning by adding new quantifiers was proposed (see, e.g., [3, 18]).

In our previous paper ([6]), we generalized Peterson's approach and syntactically proved 105 generalized syllogisms with generalized intermediate quantifiers. Recall that generalized intermediate quantifiers are expressions of a natural language, which form a transition between universal and existential generalized quantifiers (cf., for example,[21, 20]). Typical examples are *"Almost all children like chocolate", "Most children do not like mathematics", etc.* Other mathematical models of some generalized quantifiers were suggested by several authors, for example Hájek, Pereira and others [5, 4, 14, 15]. Later, in [8], we proposed a general structure of generalized syllogisms, while the possibility of syllogisms with more premises was discussed.

The main objective of this paper is to continue in studying of the theory of syllogistic reasoning and to analyze new syllogisms.

2 Preliminaries

In this section, we will briefly recall a few main concepts of the Łukasiewicz fuzzy type theory (Ł-FTT) and of the theory of evaluative linguistic expressions. The reader can find details in several papers [7, 10, 11].

2.1 Fuzzy type theory

The basic syntactical objects of Ł-FTT are classical (cf. [1]), namely the concepts of type and formula. The atomic types are ϵ (elements) and o (truth values). General types are denoted by Greek letters $\alpha, \beta, \ldots$. We will omit the type whenever it is clear from the context. The set of all types is denoted by *Types*. The (meta-)symbol ":=" used below means "is defined by".

The *language* consists of variables $x_\alpha, \ldots$, special constants $c_\alpha, \ldots$ ($\alpha \in$ *Types*), the symbol λ, and parentheses. The connectives (which are special constants) are *fuzzy equality/equivalence* $\equiv$, *conjunction* $\wedge$, *implication* $\Rightarrow$, *negation* $\neg$, *strong conjunction* $\&$, *strong disjunction* ∇, *disjunction* $\vee$, and *delta* Δ.

Ł-FTT has 17 logical axioms (all of them can be found in [10]) and two inference rules while rules of modus ponens and generalization are derived rules in Ł-FTT. We recall that by $T \vdash A_o$ we mean that A_o is provable in T. We will sometimes also use the short $A_o \not\equiv B_o$ instead of $\neg(A_o \equiv B_o)$.

The truth values form an MV_Δ-algebra (see [2, 13]). Its special case is the standard Łukasiewicz MV_Δ-algebra

$$\mathscr{L} = \langle [0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1, \Delta \rangle \tag{1}$$

where

$$\wedge = \text{minimum}, \qquad \vee = \text{maximum},$$

$$a \otimes b = 0 \vee (a + b - 1), \qquad a \to b = 1 \wedge (1 - a + b),$$
$$\neg a = a \to 0 = 1 - a, \qquad \Delta(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Łukasiewicz disjunction is ∇ interpreted by $a \oplus b = 1 \wedge (a + b)$.

Let J be a language of Ł-FTT and $(M_\alpha)_{\alpha \in Types}$ be a system of sets called *basic frame* such that M_o, M_ϵ are sets and for each $\alpha, \beta \in Types$, $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$, i.e., it is a set of functions from M_α to M_β. The *general frame* is a tuple

$$\mathscr{M} = \langle (M_\alpha, \mathring{=}_\alpha)_{\alpha \in Types}, \mathscr{L}_\Delta \rangle \tag{2}$$

so that the following holds:

(i) The $\mathscr{L}_\Delta$ is a structure of truth values that is a linearly ordered MV_Δ-algebra. We put $M_o = L$ and assume that the set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathscr{L}_\Delta$.

(ii) $\mathring{=}_\alpha$ is a fuzzy equality on M_α, i.e., $\mathring{=}_\alpha \in M_{(o\alpha)\alpha}$ for every $\alpha \in Types$.

A frame $\mathscr{M}$ is a general *model* of T ($\mathscr{M} \models T$) if it is a general frame and all axioms of T are true in the degree $\mathbf{1}$ in $\mathscr{M}$. If A_o is true in the degree $\mathbf{1}$ in all models of T then we write $T \models A_o$.

Interpretation of formulas in a frame $\mathscr{M}$ is defined w.r.t. an assignment p of elements from $\mathscr{M}$ to variables. Namely, p is a function from the set of all variables of the language J to elements from $\mathscr{M}$ in keeping with the corresponding types. The set of all assignments over $\mathscr{M}$ is denoted by $\mathrm{Asg}(\mathscr{M})$.

Theorem 1 ([10]) *(a) A theory T is consistent iff it has a general model $\mathscr{M}$.*

(b) For every theory T and a formula A_o

$$T \vdash A_o \quad \textit{iff} \quad T \models A_o.$$

2.2 Theory of evaluative linguistic expressions

Evaluative linguistic expressions are expressions of a natural language such as *small, medium, big, very short, more or less deep, quite roughly strong, extremely high*, etc. In the model of intermediate quantifiers, we consider evaluative expressions in the following simple form:

$$\langle \text{linguistic hedge} \rangle \langle \text{TE-adjective} \rangle. \tag{3}$$

The model of the semantics of expressions (3) is formulated in the special theory T^{Ev} of FTT. Its language J^{Ev} has the following special symbols:

(i) The constants $\top, \bot \in Form_o$ for truth and falsity and $\dagger \in Form_o$ representing the middle truth value.

(ii) A special constant $\sim \in Form_{(oo)o}$ for an additional fuzzy equality on the set of truth values L.

(iii) A set of special constants $\boldsymbol{\nu}, \ldots \in Form_{oo}$ for linguistic hedges and a set of triples of additional constants $\mathbf{a}_{\boldsymbol{\nu}}, \mathbf{b}_{\boldsymbol{\nu}}, \mathbf{c}_{\boldsymbol{\nu}}, \ldots \in Form_o$ where each triple is associated with one hedge $\boldsymbol{\nu}$.

For the theory of evaluative expressions, it is the concept of a linguistic context w that is important. In a model, it is a function $\mathscr{M}(w) : L \to M_\alpha$ determining an interval $[v_L = \mathscr{M}(w\bot), v_S = \mathscr{M}(w\dagger)] \cup [v_S = \mathscr{M}(w\dagger), v_R = \mathscr{M}(w\top)]$.*) We usually interpret the context by the triple of elements $w = \langle v_L, v_S, v_R \rangle$ representing the smallest, typically medium and the largest thinkable values, respectively.

In T^{Ev}, we will use the following constants representing special hedges: *Ex*, *Si*, *Ve*, *ML*, *Ro*, *QR*, *VR* $\in Form_{oo}$. They construe the linguistic hedges (*extremely, significantly, very, more-or-less, roughly, quite roughly, very roughly*, respectively)†). A special linguistic hedge that will be used below is the *empty hedge* $\bar{\boldsymbol{\nu}}$. We take this hedge as present when modeling the meaning of the fundamental evaluative trichotomy "small, medium, big" because we can take, e.g., "small" as "⟨empty hedge⟩*small*".

3 Theory of intermediate quantifiers

The model of intermediate quantifiers is based on the concept of evaluative linguistic expressions whose theory is the main constituent of the fuzzy natural logic. Motivation, fundamental assumptions and the formalization of their theory are in detail presented in [11]. Because the paper is limited by number of pages, we immediately introduce definitions of intermediate quantifiers (details can be found in [6]).

For the definition of the intermediate quantifier, we need a special operation "cut of a fuzzy set" for the given fuzzy sets $y, z \in Form_{o\alpha}$:

$$y|z \equiv \lambda x_\alpha \cdot zx \,\&\, \Delta(\Upsilon(zx) \Rightarrow (yx \equiv zx)).$$

Lemma 1 *Let $\mathscr{M}$ be a model and p an assignment such that $B = \mathscr{M}_p(y) \subsetneq M_\alpha$, $Z = \mathscr{M}_p(z) \subsetneq M_\alpha$. Then for any $m \in M_\alpha$*

$$\mathscr{M}_p(y|z)(m) = (B|Z)(m) = \begin{cases} B(m), & \text{if } B(m) = Z(m), \\ 0 & \text{otherwise.} \end{cases}$$

One can see that the operation $B|Z$ "cuts" B by taking only those $m \in M_\alpha$ from the fuzzy set B whose membership $B(m)$ is equal to $Z(m)$, otherwise $(B|Z)(m) = 0$.

*) We write the interval $[v_L, v_R]$ as the union of intervals $[v_L, v_S] \cup [v_S, v_R]$ to emphasize the role of the middle point v_S (typically medium).

†) Of course, this is only a tentative list that can be extended, if necessary.

If there is no such element then $B|Z = \emptyset$. We can thus take various fuzzy sets Z to "pick up proper elements" from B.

Definition 1 *Let T^{IQ} be a theory containing intermediate quantifiers w.r.t. a set of types $\mathscr{S}$. Let $Ev \in Form_{oo}$ be a formula representing some evaluative linguistic expression. Finally, let $z \in Form_{o\alpha}$, $x \in Form_{\alpha}$ be variables and $A, B \in Form_{o\alpha}$ be formulas and $T^{IQ} \vdash \mathbf{M}_{o(o\alpha)}B$, $\alpha \in \mathscr{S}$. An intermediate quantifier of type $\langle 1,1\rangle$ is one of the following formulas:*

$$(Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)[(\forall x)((B|z)\, x \Rightarrow Ax) \wedge Ev((\mu B)(B|z))], \tag{4}$$

$$(Q^{\exists}_{Ev}\, x)(B, A) \equiv (\exists z)[(\exists x)((B|z)x \wedge Ax) \wedge Ev((\mu B)(B|z))]. \tag{5}$$

Either of the quantifiers (4) or (5) construes the sentence

$$\langle \textit{Quantifier} \rangle\ B\textit{'s are } A.$$

An intermediate quantifier of type $\langle 1,1\rangle$ with presupposition is one of the following formulas:

$$({}^{*}Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)[(\exists x)(B|z)x\, \&(\forall x)((B|z)\, x \Rightarrow Ax) \wedge Ev((\mu B)(B|z))], \tag{6}$$

$$({}^{*}Q^{\exists}_{Ev}\, x)(B, A) \equiv (\exists z)[\neg(\exists x)Bx\, \nabla((\exists x)((B|z)x \wedge Ax) \wedge Ev((\mu B)(B|z)))]. \tag{7}$$

The formula $B_{o\alpha}$ in (i)–(iii) represents a universe of quantification.

If we replace the metavariable Ev in (4)–(7) by a formula representing a specific evaluative linguistic expression, we obtain a definition of a concrete intermediate quantifier. As a special case, the four basic quantifiers *all, almost all, most* are obtained from (4), (6), using the evaluative expressions *the biggest (Bi Δ), extremely big (Bi Ex), very big (Bi Ve)*, respectively. The quantifier *many* is obtained from (4) using the expression *not small ($\neg(Sm\, \bar{\nu})$)* (for details see [6, 7]). Using *similarly small (Sm Si)* we define the quantifier "A few" and by *very small (Sm Ve)* we can introduce the quantifier "Several". Details can be found in [12].

Namely, we will consider the following quantifiers:

(**A**) "All B's are A": $(Q^{\forall}_{Bi\,\Delta}x)(B, A)$ (**E**) "No B's is A": $(Q^{\forall}_{Bi\,\Delta}x)(B, \neg A)$

(**P**) "Almost all B's are A": $(Q^{\forall}_{Bi\ Ex}x)(B, A)$ (**B**) "Almost all B's are not A": $(Q^{\forall}_{Bi\ Ex}x)(B, \neg A)$

(**T**) "Most B's are A": $(Q^{\forall}_{Bi\ Ve}x)(B, A)$ (**D**) "Most B's are not A": $(Q^{\forall}_{Bi\ Ve}x)(B, \neg A)$

(**F**) "A few B's are A": $(Q^{\forall}_{Sm\ Si}x)(B, A)$ (**V**) "A few B's are not A": $(Q^{\forall}_{Sm\ Si}x)(B, \neg A)$

(**S**) "Several B's are A": $(Q^{\forall}_{Sm\ Ve}x)(B, A)$ (**Z**) "Several B's are not A": $(Q^{\forall}_{Sm\ Ve}x)(B, \neg A)$

(**K**) "Many B's are A": $(Q^{\forall}_{\neg\ Sm}x)(B, A)$ (**G**) "Many B's are not A": $(Q^{\forall}_{\neg\ Sm}x)(B, \neg A)$

(**I**) "Some B's are A": $(Q^{\exists}_{Bi\ \mathbf{\Delta}}x)(B, A)$ (**O**) "Some B's are not A": $(Q^{\exists}_{Bi\ \mathbf{\Delta}}x)(B, \neg A)$

Below we introduce the theorem which explains the monotonicity property of five basic quantifiers.

Theorem 2 (Monotonicity[6]) *(a)* $T^{IQ} \vdash \mathbf{A} \Rightarrow \mathbf{P}$, $T^{IQ} \vdash \mathbf{P} \Rightarrow \mathbf{T}$, $T^{IQ} \vdash \mathbf{T} \Rightarrow \mathbf{K}$.

(b) $T^{IQ} \vdash \mathbf{E} \Rightarrow \mathbf{B}$, $T^{IQ} \vdash \mathbf{B} \Rightarrow \mathbf{D}$, $T^{IQ} \vdash \mathbf{D} \Rightarrow \mathbf{G}$.

The monotonicity property for new quantifiers was introduced and proved in [12]. Recall that the special axiom is assumed (see Subsection 4.1 in [12])].

Theorem 3 (Monotonicity [12]) *Let $B_{o\alpha}, A_{o\alpha}$ be formulas. Then the following is provable in T^{IQ}:*

(a) $T^{IQ} \vdash (\mathbf{S}) \Rightarrow (\mathbf{I})$,

(b) $T^{IQ} \vdash (\mathbf{F}) \Rightarrow (\mathbf{S})$.

(c) Let $T^{IQ} \vdash \mathbf{SQ}_{o(o\alpha)}\, B_{o\alpha}$. *Then* $T^{IQ} \vdash Q^{\forall}_{\neg\ Sm\ \bar{\boldsymbol{\nu}}}x)(B, A) \Rightarrow Q^{\forall}_{+\ Sm\ Si}x)(B, A)$ *(i.e.,* $T^{IQ} \vdash (\mathbf{K}) \Rightarrow (\mathbf{F})$*).*

4 Generalized Peterson's syllogisms with "A few" and "Several"

The categorical syllogism is a special kind of a logical argument in which the conclusion is inferred from two premises: the major premise (first) and minor premise (second). The intermediate syllogism is obtained from any traditional syllogism when replacing one or more of its formulas by formulas containing intermediate quantifiers. As we mentioned above, the 105 (24 Aristotle's and 81 Peterson's) generalized Peterson's syllogisms with two premises were syntactically proved in [6]. The construction was based on the assumption of one middle formula and four corresponding figures. The structure of all the generalized Peterson's syllogisms can be found in [8].

In general, we can define syllogism as follows. Let P_1, P_2, C be quantifiers of a certain type. By *syllogism*, we understand a triple $\langle P_1, P_2, C\rangle$ where P_1 is a *major premise*, P_2 a *minor premise* and C is a *conclusion*.

We say that the syllogism $\langle P_1, P_2, C\rangle$ is *strongly valid* in a theory T if P_1, P_2, C are formulas of the language $J(T)$ and $T \vdash P_1 \,\&\, P_2 \Rightarrow C$, or equivalently, if $T \vdash P_1 \Rightarrow (P_2 \Rightarrow C)$.

Classical theory of syllogisms deals with quantifiers of type $\langle 1, 1, \rangle$ divided into four figures. Let Q_1, Q_2, Q_3 be intermediate quantifiers of type $\langle 1, 1\rangle$ and $X, Y, M \in Form_{o\alpha}$ be formulas representing properties. Then the following figures represent basic syllogisms:

Figure I	**Figure II**	**Figure III**	**Figure IV**
$Q_1\ M$ is Y	$Q_1\ Y$ is M	$Q_1\ M$ is Y	$Q_1\ Y$ is M
$Q_2\ X$ is M	$Q_2\ X$ is M	$Q_2\ M$ is X	$Q_2\ M$ is X
$Q_3\ X$ is Y	$Q_3\ X$ is Y	$Q_3\ X$ is Y	$Q_3\ X$ is Y

where the first line in each figure is the major premise P_1, the second line is the minor premise P_2 and the third line is the conclusion C. If $Q_1, Q_2, Q_3 \in \{\forall, \exists\}$ then the above syllogisms are classical.

To simplify the notation, we will write syllogism of the corresponding figure as P_1P_2C-I, ..., P_1P_2C-IV.

4.1 Figure-I

Theorem 4 *Strong validity of syllogisms* **AAA**-*I,* **APP**-*I,* **ATT**-*I,* **AKK**-*I,* **AFF**-*I and* **ASS**-*I in* T^{IQ} *implies strong validity of the syllogisms summarized below:*

Figura-I: Positive generalized syllogisms

AAA						
AAP	**APP**					
AAT	**APT**	**ATT**				
AAK	**APK**	**ATK**	**AKK**			
AAF	**APF**	**ATF**	**AKF**	**AFF**		
AAS	**APS**	**ATS**	**AKS**	**AFS**	**AFF**	
A*AI	**A*PI**	**A*TI**	**A*KI**	**A*FI**	**A*SI**	**AII**

Proof 1 *The validity of the syllogisms* ***AAA, APP, ATT, AKK, AFF, ASS*** *can be obtained using Theorem 11 from [6]. The proofs of syllogisms in the first column can be proved using the validity of the syllogism* ***AAA****-I applying Theorem 2. Analogously we continue with other columns. The novelty of this proof is to*

apply new monotonicity Theorem 3 for the quantifiers "A few" and "Several" for the verification of the validity of syllogisms ***AAF***, ***APF***, ***ATF***, ***AKF****-I.*

We continue with negative generalized syllogisms of Figure-I.

Theorem 5 *Strong validity of syllogisms* **AAA***-I,* **APP***-I,* **ATT***-I,* **AKK***-I,* **AFF***-I and* **ASS***-I in* T^{IQ} *implies strong validity of the syllogisms summarized below:*

Figura-I: Negative generalized syllogisms

EAE						
EAB	**EPB**					
EAD	**EPD**	**ETD**				
EAG	**EPG**	**ETG**	**EKG**			
EAV	**EPV**	**ETV**	**EKV**	**EFV**		
EAZ	**EPZ**	**ETZ**	**EKZ**	**EFZ**	**ESZ**	
E*AO	**E*PO**	**E*TO**	**E*KO**	**E*FO**	**E*SO**	**EIO**

Proof 2 *Analogously as in the previous theorem.*

5 Conclusion

In this article, we continued to study the theory of syllogistic reasoning. We focused on generalized syllogisms with new intermediate quantifiers "A few" and "Several". We syntactically proved generalized syllogisms of Figure-I. Recall that all the presented syllogisms are bordered by Aristotle's syllogisms, which means that all the generalized syllogisms have been caused by weakening or strengthening of the conclusion. Other generalized syllogisms of other figures will be analyzed in the prepared paper.

6 Acknowledgement

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Firefly Algorithm for Hyper-Parameter Optimization of L^2-Distance Estimation Models

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Abstract

When analyzing uncertain data expressed by a set of data points, calculating the density difference between any two sets of data points is an important task. Least squares density difference (LSDD) estimator which uses a non-parametric model approximates the true density difference accurately. In the previous method, it searches for the parameters of the model by using the grid search. However, there is no guarantee that the obtained parameters by the grid search are optimal, we apply a firefly algorithm which is suitable to the multimodal optimization problem for the parameter search. Numerical experiments are conducted to show that the LSDD estimator by using the firefly algorithm approximate the true density differences and L2-distances more accurately than the grid search.

1 Introduction

It is usually difficult to consistently obtain accurate measurement values as the noise cannot removed. Thus, it is not necessary to measure once, instead, it is better to repeatedly conduct the measurement and obtain a distribution fo the measured values.

As stated above, there is uncertainty in the measurement due to the instability of measuring devices and measurement environments as well as the instability of the objects to be measured. When analyzing such measured values with uncertainty, one way is to use the whole distribution of the measured values instead of the representative values such as averages and variances. For example, in order to

analyze the difference between two objects with such uncertainty measurements, it is necessary to calculate the difference between the two distributions obtained through the measurements of the two objects.

One way is to calculate the representative values of a distribution such as the average and the median to calculate the difference between two distributions. However, such representative values cannot consider the variance in the distributions. There is another way to estimate the probability density function and the difference between the two corresponding probability density functions are calculated. However, this method would suffer from two errors: One is the error in the estimation of the probability density functions, and the other is in the estimation of the difference between the estimated probability density functions.

This paper considers a direct estimation method for estimating the difference between two distributions without such two-step estimations. This method is called Least Squared Density Difference method (LSDD) [1]. It was shown in [1] that the LSDD estimation method produces the estimation with less error than two-step estimation method. As the application of the LSDD estimation method, two-sample test [2], the estimation of the class prior under the condition of varying class balance [3], and clustering of probability distributions [4].

In the LSDD estimation method, it is necessary to find the appropriate parameters of non-parametric models that help approximate the true density difference. A common approach for finding the appropriate parameter values is called Grid Search (GS) where all combinations of the candidate parameters are tried and pick up the optimal parameters that produce the best performance. However, as GS only finds the best parameter set among the candidate sets, the obtained parameter set might not be the best among all the possible parameter sets. In this paper, we try to overcome this problem by using a hyper heuristics called firefly algorithm [5].

A series of computational experiments are conducted to compare the performance of the parameter search algorithms between GS and the firefly algorithm. The performance of the search algorithms are examined by the difference between the true density difference and the calculated difference by the search algorithms.

2 Least Squared Density Difference (LSDD) Estimation

Now, let us assume that there are two sets (called distribution hereafter) of d-dimensional points as follows:

$$X = \{\vec{x}_i\}_{i=1}^{m} \sim p(\vec{x}), \tag{1}$$

$$X' = \{\vec{x'}_j\}_{j=1}^{m} \sim p'(\vec{x}), \tag{2}$$

where $p(\vec{x})$ and $p(\vec{x'})$ are the probability density functions for X and X', respectively. It is assumed that the probability density functions are not known a priori.

In this case, the distance between $p(\vec{x})$ and $p(\vec{x'})$ is calculated in terms of L^2 distance, which is defined as follows:

$$L^2\left(p(\vec{x}), p'(\vec{x'})\right) = \int \left(p(\vec{x}) - p'(\vec{x})\right)^2 d\vec{x}. \tag{3}$$

Although it is hardly possible to calculate the exact value of the L^2 distance, it is possible to estimate it from the two distributions X and X'. The LSDD estimation minimizes the error between the estimated value and the true density difference. Specifically, the following squared error is minimized by the LSDD estimation:

$$\text{Minimize} \int \left\{g(\vec{x}) - \left(p(\vec{x}) - p'(\vec{x})\right)^2\right\} d\vec{x}. \tag{4}$$

In this paper, the following Gaussian kernel model is used to represent the density difference model:

$$g(\vec{x}) = \sum_{l=1}^{n+m} \theta_l \exp\left\{-\frac{||\vec{x} - \vec{c}_l||^2}{2\sigma^2}\right\}, \tag{5}$$

where $(\vec{c}_1, \ldots, \vec{c}_n, \vec{c}_{n+1}, \ldots, \vec{c}_{n+m}) = (\vec{x}_1, \ldots, \vec{x}_n, \vec{x}_{n+1}, \ldots, \vec{x}_m)$ is the center of the Gaussian kernel, $\theta = (\theta_1, \ldots, \theta_{n+m})$ is a set of the model parameters, and σ is the width of the kernel. The optimal value of the model parameters θ^* can be obtained by the following equation:

$$\begin{aligned}
\theta^* &= \arg\min_\theta \int \left\{g(\vec{x}) - \left(p(\vec{x}) - p'(\vec{x})\right)\right\}^2 d\vec{x} \\
&= \arg\min_\theta \left[\theta^T H \theta - 2h\theta\right] \\
&= H^1 h,
\end{aligned} \tag{6}$$

where H is a matrix of $(n+m) \times (n+m)$ elements and h is an $(n+m)$-dimensional vector. They are defined as follows:

$$\begin{aligned}
H_{ll'} &= \int exp\left(-\frac{||\vec{x} - \vec{c}_l||^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{||\vec{x} - \vec{c}_{l'}||^2}{2\sigma^2}\right) d\vec{x} \\
&= (\pi\sigma^2)^{\frac{d}{2}} \cdot \exp\left(-\frac{||\vec{c}_l - \vec{c}_{l'}||}{4\sigma^2}\right),
\end{aligned} \tag{7}$$

$$h_l = \int \exp\left(-\frac{||\vec{x} - \vec{c}_l||^2}{2\sigma^2}\right) \cdot p(\vec{x}) d\vec{x} - \int \exp\left(-\frac{||\vec{x'} - \vec{c}_{l'}||^2}{2\sigma^2}\right) p'(\vec{x'}) d\vec{x'}. \tag{8}$$

The calculation of h_l is not possible because the probability density functions in (8) are not known. Thus, h in (6) is replaced with its estimate $\hat{h}$ and to reformulate the optimization with the addition of a l_2-regularization term. The objective function of the re-formulated optimization problem is written as follows:

$$\hat{\theta} = \arg\min_\theta \left[\theta^T H \theta - 2\hat{h}^T \theta + \lambda \theta^T \theta\right], \tag{9}$$

where $\lambda(\leq 0)$ is a constant for the regularization and $\hat{h}$ is an $(n+m)$-dimensional vector which is defined by the following:

$$\hat{h}_l = \frac{1}{n}\sum_{i=1}^{n} \exp\left(-\frac{||\vec{x}_i - \vec{c}_l||^2}{2\sigma^2}\right) - \frac{1}{m}\sum_{j=1}^{m} \exp\left(-\frac{||\vec{x'}_j - \vec{c}_l||^2}{2\sigma^2}\right) \tag{10}$$

From the above, the solution (i.e., the optimal value for θ) is as follows:

$$\hat{\theta} = (H + \lambda I)^{-1}\hat{h}. \tag{11}$$

The precision of the LSDD estimate depends on the specification of the parameters such as the kernel width and the regularization constant. Usually, the optimal values for the kernel width and the regularization constant are obtained through the cross-validation using the distributions X and X'. In the cross-validation, the distributions X and X' are divided into T subsets. The T subsets are used one of the following two usages: One is to model the density difference model and the other is to estimate the hold-out error CV which is defined by the following:

$$CV^t = \int \hat{g}_t(\vec{x})^2 d\vec{x} - \frac{2}{|X_t|}\sum_{\vec{x}\in X_t} \hat{g}_t(\vec{x}) + \frac{2}{|X'_t|}\sum_{\vec{x'}\in X'_t} \hat{g}_t(\vec{x'}), \quad t = 1, \ldots, T. \tag{12}$$

The following average hold-out error is calculated to pick up the optimal parameter that produces the minimum one:

$$CV = \frac{1}{T}\sum_{t=1}^{T} CV^t. \tag{13}$$

The estimated density difference with the optimal parameters is obtained by the following equation:

$$\hat{g}(\vec{x}) = \sum_{l=1}^{n+m} \hat{\theta}_l \cdot \exp\left(-\frac{||\vec{x} - \vec{c}_l||^2}{2\sigma^2}\right). \tag{14}$$

Furthermore, the L^2 difference is estimated considering the regularization bias as follows:

$$\hat{L}^2\left(p(\vec{x}), p'(\vec{x})\right) = 2\hat{h}^T\hat{\theta} - \hat{\theta}^T H\hat{\theta}. \tag{15}$$

In this paper, we use grid search (GS) and Firefly algorithm (FA) to search for the optimal parameters of the kernel width and the regularization constant that minimizes the average hold-out error CV in (13).

3 Firefly Algorithm (FA)

Firefly algorithm is one of nature-inspired meta-heuristics with a swarm of fireflies. A firefly has a function of lightning on the tail. In the firefly algorithm, the position

of a firefly represents the set of the parameters to be optimized and the intensity of the firefly is assumed to be proportional to the quality function in terms of an objective function to be optimized. The following assumptions are made in the firefly algoirthm:

1. There is no sexuality in the fireflies (i.e., no male nor female).

2. A firefly is attractive when its light intensity is strong. A firefly with a weaker light intensity approaches to the one with the stronger light intensity. The attractiveness also depends on the distance between the two fireflies. That is, even if one firefly has a strong light intensity, it does not attract the one which locates far away from it.

3. The firefly with the strongest light intensity among all the fireflies moves randomly.

The procedure of the firefly algorithm with the above assumptions is shown in Algorithm 1.

Objective function: $f(\vec{x}), \vec{x} = (x_1, \ldots, x_d)$
Initial position of fireflies: $\vec{x}_i, i = 1, \ldots, M$
Light intensity of the i-th firefly: $I_i = -f(\vec{x}_i)$

```
Set t = 0;
while t < MaxGenerationt_max do
    for i = 1 to M do
        for j = 1 to M do
            if I_j > I_i then
                Move x_i toward x_j;
                Update I_i;
            end
        end
    end
    Randomly move the best firefly with the largest intensity;
end
```

Algorithm 1: Firefly algorithm.

In the firefly algorithm, there are M fireflies, each of which represents a solution of the problem to be solved. The fireflies are attracted to each other and find the position that produce the largest light intensity, which means that the parameters are the optimal in terms of the objective function. Because our optimization problem aims to minimize the objective function, the light intensity is defined as the negative value of the objective function. This allows the firefles to search for the

best position with the largest light intensity. The initial position of the fireflies are randomly determined within a pre-defined solution domain.

Let us assume that the j-th firefly is more attractive than the i-th firefly. That is, we assume that $I_j > I_i$. In this case, the i-th firefly is attracted by the j-th firefly and is moved towards. The following equation is used for updating the position of teh i-th firefly by the attraction:

$$x_{id}^{\text{new}} = x_{id}^{\text{old}} + \beta_{i,j} \cdot (x_{jd}^{\text{old}} - x_{id}^{\text{old}}) + \alpha \cdot \epsilon_i, \\ i, j = 1, \ldots, M, \quad i \neq j, \quad d = 1, \ldots, m, \tag{16}$$

where M is the number of the fireflies in the swarm, m is the total number of parameters to be searched for, ϵ_i is a uniform random value for the i-th firefly within an interval $[-0.5, 0.5]$, α is a control value that decides the degree of the influence of the random value (i.e., ϵ_i). $\beta_{i,j}$ is a ratio of the i-th firefly's attractiveness to the j-th firefly's one and is defined as follows:

$$\beta_{i,j} = \beta_0 \cdot e^{-\gamma \cdot r_{i,j}^2}, \tag{17}$$

where β_0 is a positive constant value that represents the default attractiveness when $r_{i,j} = 0$, γ is an optical-absorption coefficient, and $r_{i,j}$ is the distance between the i-th firefly and the j-th firefly and is defined as the Euclidean distance between them.

The best firefly with the largest light intensity is randomly moved by the following equation:

$$x_{\text{best},d}^{\text{new}} = x_{\text{best},d}^{\text{old}} + \alpha \cdot \epsilon_i, d = 1, \ldots, m. \tag{18}$$

If the updated position of the best firefly by the above equation leads to a lesser light intensity, the best firefly is moved back to its original position.

4 Computational Experiments

4.1 Experimental Settings

The following probability density functions are used for the computational experiments of this paper:

$$p(\vec{x}) = N\left(\vec{x}; (a, 0, \ldots, 0)^T, (4\pi)^{-1} I_d\right), \tag{19}$$

$$p'(\vec{x}) = N\left(\vec{x}; (0, 0, \ldots, 0)^T, (4\pi)^{-1} I_d\right), \tag{20}$$

where$N(\vec{x}; \mu, \sum)$ represents a multi-dimensional normal distribution with the mean μ and the variance-covariance matrix $\sum$, and I_d is a d-dimensional identity matrix. From each of these probability density functions, five data sets are generated. These sampled distributions are used to calculate the estimated density difference by using L^2 distance. The grand truth of the density difference is obtained by the average

over all the combination of the two distribtutions from different probability density functions.

In the search methods using the firefly algorithm (FA) and the grid search (GS), the performance is measured by five-hold cross validation method. In the grid search, the best parameter set among all possible combination of the following candidates of the kernel width σ and the regularization constant λ is used as the search result:

$$\sigma \in \left\{10^{-2}, 10^{-1.5}, 10^{-1}, 10^{-0.5}, 10^{0}\right\}, \tag{21}$$

$$\lambda \in \left\{10^{-1}, 10^{-0.5}, 10^{0}, 10^{0.5}, 10^{1}\right\}. \tag{22}$$

In the firefly algorithm, the position of a firefly is defined as (σ, λ). That is, the search space is spanned by the kernel width σ and the regularization constant λ. The firefly algorithm searches for the best parameters of σ and λ that minimizes the objective function. The fireflies in the swarm is initialized so that each firefly is located on the grid points in the grid search. The maximum number of iterations is set to 200, and the other parameters in the firefly algorithm are set to $(\alpha, \beta_0, \gamma) = (0.2, 1.0, 1.0)$.

4.2 Experimental Results

We first compare the search performance of the firefly algorithm and the grid search algorithm for an experimental settings of the sample size $n = m = 200$, and the dimensionality $d = 1$. The search performance is measured by the squared error between the ground truth density difference and the one obtained from each algorithm. We show the obtained squared error by each algorithm when $a = 0$ in Fig. 1. From Fig. 1, it can be seen that the squared error by the firefly algorithm is smaller than the that by the grid search.

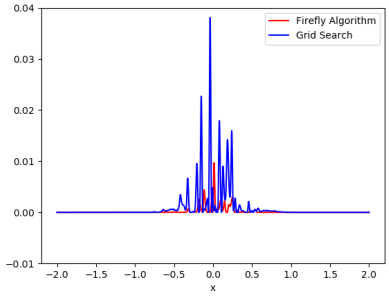

Figure 1: Squared error between the estimated density difference and its true value ($\alpha = 0$).

Next, we show the squared error between the true density difference and the estimated one by the search algorithms when $\alpha = 0.5$ in Fig. 2. We can see from

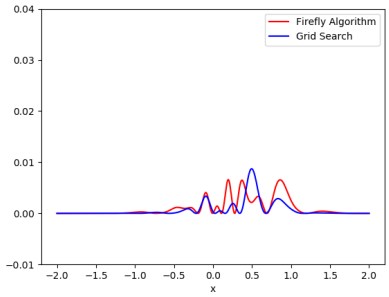

Figure 2: Squared error between the estimated density difference and its true value ($\alpha = 0.5$).

Fig. 2 that the search performance of the firefly algorithm and the grid search are almost the same.

Now, we compare the search performance between the firefly algorithm and the grid search algorithm with the following experimental settings: Sample sizes $n = m = 200$, the dimensionality $d = 1, 2, 3, 4, 5$. The search performance is measured by the difference between the true L^2 distance and the estimated L^2 distance by the search algorithms. Figure 3 shows the squared error between the true density difference and the estimated density difference obtained by the search algorithms when $a = 0$. We can see from Fig.3 that the squared error by the firefly algorithm is smaller than that by the grid search.

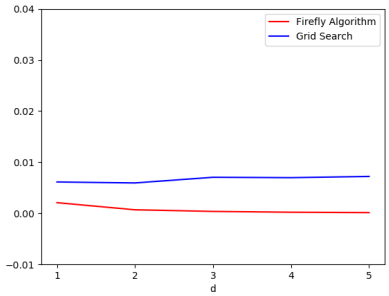

Figure 3: Squared error between the estimated density difference and its true L^2 distance ($\alpha = 0$).

The squared error between the true density difference and the estimated density difference by the search algorithms when $a = 0.5$ is shown in Fig. 4. From Fig.4, we can see that the error by the firefly algorithm is smaller than that by the grid search.

From the all experiments explained above, it is shown that the search performance by the firefly algorithm is better than the grid search.

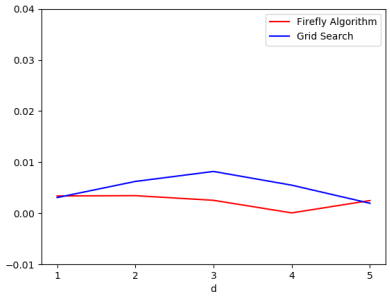

Figure 4: Squared error between the estimated density difference and its true L^2 distance ($\alpha = 0.5$).

5 Conclusions

This paper examined the performance of the firefly algorithm for finding the optimal parameters that lead the minimum objective function. The task of the search algorithms is to find the minimum density difference between the true density difference and the estimated density difference. The density difference was estimated by L^2 distance. Firefly algorithm and grid search are employed in this research. Through the computational experiments, it was shown that the firefly algorithm perform better than the grid search.

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On the optimal drawings of the products of special graphs

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Abstract

Finding minimum necessary intersections in graph representations is useful in many areas, especially automated graph drawings and VLSI-layouts. A drawing of the graph G with the vertex set V and the edge set E is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. The *crossing number* $\mathrm{cr}(G)$ of the graph G is defined as the minimal number of pairwise intersections of nonadjacent edges in any drawing of G in the plane. It is well known that the problem of determination of the crossing numbers of graphs is NP-complete [5] and it remains NP-hard even for cubic graphs [6]. The exact value of the crossing number is known only for few classes of graphs, mainly with regular structure such as various products of graphs.

The aim of this article is to extend known results and determine the exact value of the crossing number of the products of the special graphs on six vertices with paths.

1 Introduction

The problem of reducing the number of crossings is a classical and moreover very difficult problem. It is studied not only in the graph theory, but also by computer scientists, especially automated graph drawings, improved the readability of hierarchical structures and the most prominent areas - VLSI-layouts. The visualized graph should be easy to read and understand. For the understandability of graph drawings, the reducing of crossings is by far the most important.

Let G be a simple graph with vertex set V and edge set E. A drawing of G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. For simplicity, we assume that in a drawing (a) no edge passes through any

vertex other than its end-points, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point. The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with vertex set $V(G)$ and edge set $E(G)$ is defined as the minimum possible number of edge crossings in a good drawing of G in the plane. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross each other. Let G_1 and G_2 be simple graphs with vertex sets $V(G_1)$ and $V(G_2)$, and edge sets $E(G_1)$ and $E(G_2)$, respectively. The Cartesian product $G_1\Box G_2$ of the graphs G_1 and G_2 has vertex set $V(G_1\Box G_2) = V(G_1) \times V(G_2)$ and two vertices (u, u') and (v, v') are adjacent in $G_1\Box G_2$ if and only if either $u = v$ and u' is adjacent with v' in G_2, or $u' = v'$ and u is adjacent with v in G_1. Let P_n and C_n be the path and the cycle on n edges, respectively, and S_n be the star isomorphic to $K_{1,n}$. In the proofs of the paper, we will often use the term "region" also in nonplanar drawings. In this case, crossings are considered to be vertices of the "map".

The exact values of the crossing numbers are known only for some graphs or some families of graphs, mainly with regular structure such as various products of graphs. Specially for join product it was proved the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle in [11] and for join product of all graphs G of order at most four with paths and cycles in [16]. Moreover, the crossing numbers of join product the discrete graphs, paths and cycles with some graphs of order five and six are given in [12], [13], [15], [17], [18], [19], [20], [21], [22], [23] and [24].

Among the products of graphs, the Cartesian product has received great attention in the mathematical publications. In [7], [8] and [9], the crossing numbers of Cartesian products of paths, cycles and stars with all graphs of order four are given. Bokal in [1] confirmed the general conjecture for crossing numbers of Cartesian products of paths and stars formulated in [7]. The crossing numbers of Cartesian products of paths with all graphs of order five are collected in [10] and with 40 graphs of order six are collected in [14]. Determining the crossing numbers of Cartesian product of some graphs of order six, seven and eight with paths or cycles is given in [2], [3] and [4].

In the paper, these known results will be extended by determining the crossing numbers of Cartesian products of paths with other 6-vertex graphs shown in Fig. 1.

2 Preliminary results

In this section, it will be proven some lemmas, which help to give the crossing numbers of Cartesian products of paths with graph G_1.

We assume $n \geq 1$ and find it convenient to consider the graph $G_1\Box P_n$ in the following way: it has $6(n + 1)$ vertices and edges that are the edges in $n + 1$ copies of G_1^i, $i = 0, 1, \ldots, n$, and in six paths of length n. The labeling of the

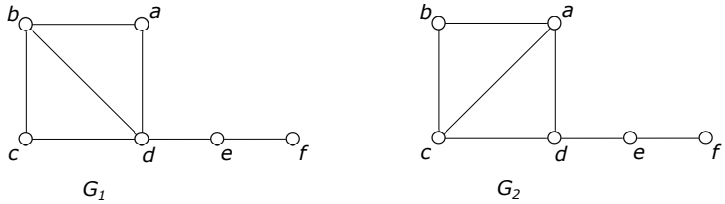

Figure 1: The graphs G_1 and G_2 on six vertices

vertices of graphs G_1 is shown in Fig. 1. Let us denote by M^i the subgraph of $G_1 \square P_n$ containing the vertices of G_1^{i-1} and G_1^i and six edges joining G_1^{i-1} to G_1^i, $i = 1, 2, \ldots, n$. Let Q^i, $i = 1, 2, \ldots, n-1$, denote the subgraph of $G_1 \square P_n$ induced by $V(G_1^{i-1}) \cup V(G_1^i) \cup V(G_1^{i+1})$. So, $Q^i = G_1^{i-1} \cup M^i \cup G_1^i \cup M^{i+1} \cup G_1^{i+1}$. Let K^i denote the subgraph of G_1^i which is isomorphic with $K_{1,1,2}$, $i = 0, 1, \ldots, n$ and let $K \square P_n$ denote the graph $K_{1,1,2} \square P_n$. By M_K^i we will denote the corresponding subgraph of M^i in $K \square P_n$.

In a good drawing D, we say that a graph K^i separates the graphs K^p and K^q (the vertices of a graph K^p) if there exists a cycle C of K^i such that K^p and K^q are contained in different components of $\mathbb{R}^2 \setminus C$.

The following results enable us to simplify the proofs in the next section.

Lemma 1 *Let D be a good drawing of the graph $K \square P_n$, $n \geq 2$, in which each of the graphs K^{i-1}, K^i, and K^{i+1}, $i = 1, 2, \ldots, n-1$, has at most two crossings on its edges. Then K^{i-1} does not separate K^i and K^{i+1}, K^{i+1} does not separate K^i and K^{i-1}, and if K^i has an internal crossing, K^i does not separate K^{i-1} and K^{i+1}.*

Proof. If K^{i-1} separates K^i and K^{i+1} (K^{i+1} separates K^i and K^{i-1}), then the subgraph K^{i-1} (K^{i+1}) is crossed by all four edges joining the separated subgraphs. This contradicts the assumption that every graph K^i has at most two crossings on its edges. It remains to show that K^i does not separate K^{i-1} and K^{i+1}, if K^i has an internal crossing. Without loss of generality, let K^{i-1} is placed inside K^i. The subdrawing of K^i induced by D divides the plane in such a way that on the boundary of every region inside K^i there are at most two vertices of K^i (see Fig. 2(b)). Thus, one of graphs K^{i-1} and M_K^i crosses K^i at least twice. So, this contradicts the assumption of at most two crossings of every graph K^i in D and we are done. □

Lemma 2 *Let D be a good drawing of the graph $K \square P_n$, $n \geq 2$, in which each of the graphs K^{i-1}, K^i, and K^{i+1}, $i = 1, 2, \ldots, n-1$, has at most two crossings on its edges and none of them separates two other. Then K^{i-1}, K^i and K^{i+1} do not cross each other.*

Proof. Assume a good drawing D of the graph $K\square P_n$, $n \geq 2$, in which each of the graphs K^{i-1}, K^i, and K^{i+1}, $i = 1, 2, \ldots, n-1$, has at most two crossings on its edges and none of them separates two other. If two of the 2-connected graphs K^{i-1}, K^i, and K^{i+1} cross, then they cross at least twice. So, none of K^{i-1}, K^i, and K^{i+1} crosses both others. Moreover, if two graphs of K^{i-1}, K^i, and K^{i+1} cross, then none of them has an internal crossing. If $\mathrm{cr}_D(K^i, K^{i-1}) \neq 0$, then the subdrawing of K^i induced by D divides the plane as shown in Fig. 2(a) and, as $\mathrm{cr}_D(K^i, K^{i+1}) = 0$, in D at least one edge of M_K^{i+1} joining K^i to K^{i+1} crosses K^i. This contradicts the assumption of at most two crossings on the edges of K^i. The same contradiction is obtained if $\mathrm{cr}_D(K^i, K^{i+1}) \neq 0$. The last possibility is that $\mathrm{cr}_D(K^{i-1}, K^{i+1}) \neq 0$. In this case K^{i-1} divides the plane as shown in Fig. 2(a) and at least one edge of M_K^i crosses K^{i-1}. This contradiction completes the proof. □

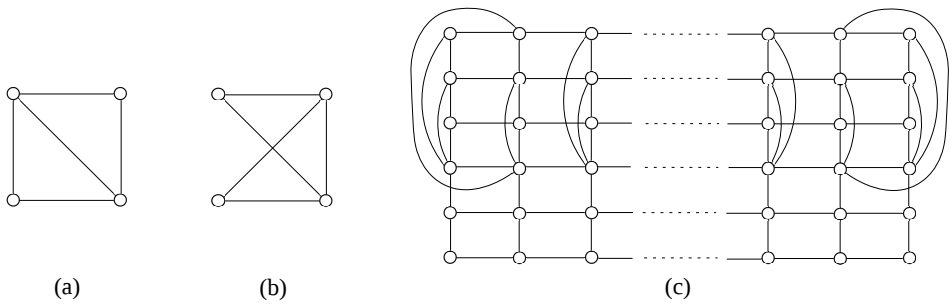

Figure 2: The unique planar drawing of K^i, drawing of K^i with internal crossing and the graph $G_1\square P_n$

Lemma 3 *Let D be a good drawing of the graph $K\square P_2$ in which every graph K^i, $i = 0, 1, 2$, has at most two crossings on its edges. If the subgraphs K^0, K^1, and K^2 do not cross each other, none of them separates two other and K^1 has an internal crossing, then* $\mathrm{cr}_D(K^1) + \mathrm{cr}_D(K^1, M_K^1 \cup M_K^2) + \mathrm{cr}_D(K^0 \cup M_K^1, K^2 \cup M_K^2) \geq 3$.

Proof. Assume that there is a good drawing D of the graph $K\square P_2$ in which the graphs K^0, K^1, and K^2 do not cross each other, none of them separates two other and K_1 has an internal crossing and that $\mathrm{cr}_D(K^1) + \mathrm{cr}_D(K^1, M_K^1 \cup M_K^2) + \mathrm{cr}_D(K^0 \cup M_K^1, K^2 \cup M_K^2) \leq 2$. Then at least one of $K^0 \cup M_K^1$ and $M_K^2 \cup K^2$ does not cross K^1. Without loss of generality, let $\mathrm{cr}_D(K^1, K^0 \cup M_K^1) = 0$. Then the subdrawing of $K^0 \cup M_K^1 \cup K^1$ induced by D divides the plane in such a way that on the boundary of every region outside K^1 there are at most two vertices of K^1 (see Fig. 3(b)). The graph K^2 does not cross an edge of the 2-connected subgraph $K^0 \cup M_K^1 \cup K^1$, otherwise $\mathrm{cr}_D(K^1) + \mathrm{cr}_D(K^1, M_K^1 \cup M_K^2) + \mathrm{cr}_D(K^0 \cup M_K^1, K^2 \cup M_K^2) \geq 3$, a contradiction. Thus, K^2 is placed in one region outside K^1. But, in this case, at least two edges of M_K^2 joining K^2 with the vertices of K^1 cross the edges of

$K^0 \cup M_K^1 \cup K^1$. So, $\mathrm{cr}_D(K^1) + \mathrm{cr}_D(K^1, M_K^1 \cup M_K^2) + \mathrm{cr}_D(K^0 \cup M_K^1, K^2 \cup M_K^2) \geq 3$ and the proof is done. □

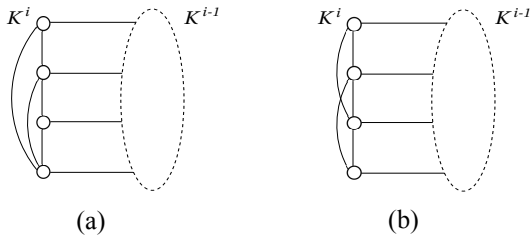

Figure 3: Two possible drawings of the graph $K^{i-1} \cup M_K^i \cup K^i$

3 The crossing number of $G_1 \square P_n$ and $G_2 \square P_n$

The graph $G_1 \square P_1$ is planar. The crossing number of the graph $G_1 \square P_2$ is two, because the graph $S_4 \square P_2$ is its subgraph and $\mathrm{cr}(S_4 \square P_2) = 2$ (see [1]). The reverse inequality $\mathrm{cr}(G_1 \square P_2) \leq 2$ one can verify by finding a suitable drawing of the graph $G_1 \square P_2$ with two crossings. In Fig. 2(c) there is the drawing of the graph $G_1 \square P_n$ with $3n - 5$ crossings. The next result is fundamental in proving that the crossing number of the graph $G_1 \square P_n$ is $3n - 5$ for $n \geq 3$.

Lemma 4 *If D is a good drawing of the graph $G_1 \square P_n$, $n \geq 3$, in which every of the subgraphs G_1^i, $i = 0, 1, 2, ..., n$, has at most two crossings on its edges, then in D there are at least $3n - 5$ crossings.*

Proof. In a drawing of the graph $G_1 \square P_n$, let us consider the following types of possible crossings on the edges of Q^i for all $i = 1, 2, ..., n - 1$:

(1) a crossing of an edge in $G_1^{i-1} \cup M^i$ with an edge in $G_1^{i+1} \cup M^{i+1}$,

(2) a crossing of an edge in $M^i \cup M^{i+1}$ with an edge in G_1^i,

(3) a crossing among the edges of G_1^i,

(4) a crossing of an edge in $G_1^{i-1} \cup G_1^{i+1}$ with an edge in G_1^i.

It is readily seen that every crossing of types (1), (2) and (3) appears in a good drawing of the graph $G_1 \square P_n$ only on the edges of one subgraph Q^i. For $i \in \{2, 3, \ldots, n-1\}$, a crossing of type (4) in Q^i between an edge of G_1^{i-1} and an edge of G_1^i appears only in Q^{i-1} as a crossing of type (4) and for $i \in \{1, 2, \ldots, n-2\}$, a crossing between an edge of G_1^{i+1} and an edge of G_1^i appears only in Q^{i+1} as a crossing of type (4).

In a good drawing of $G_1 \square P_n$, we define the force $f(Q^i)$ of Q^i in the following way: every crossing of type (1), (2), and (3) contributes the value 1 to $f(Q^i)$ and every crossing of type (4) contributes the value $\frac{1}{2}$ to $f(Q^i)$ (and $\frac{1}{2}$ to Q^{i-1} or $\frac{1}{2}$ to Q^{i+1}). The total force of the drawing is the sum of $f(Q^i)$. It is easy to see that the number of crossings in the drawing is not less than the total force of the drawing. So, the aim of this proof is to show that if each of the subgraphs G_1^i, $i = 0, 1, 2, \ldots, n$, has at most two crossings on its edges, then $f(Q^i) \geq 3$ for all $i = 2, 3, \ldots, n-2$ and $f(Q^i) \geq 2$ for $i = 1$ and $i = n-1$.

Consider now the good drawing D of $G_1 \square P_n$ assumed in Lemma 4. First, we prove that $f(Q^i) \geq 3$ for all $i = 2, 3, \ldots, n-2$. By Lemma 1, in every subdrawing $D(Q^i)$ of the subgraph Q^i induced by D, $i = 1, 2, \ldots, n-1$, K^{i-1} does not separate K^i and K^{i+1}, K^{i+1} does not separate K^i and K^{i-1}, and if K^i has an internal crossing, K^i does not separate K^{i-1} and K^{i+1}. It remains to prove that K^i does not separate K^{i-1} and K^{i+1}, if K^i has the planar drawing. Without loss of generality, let K^{i-1} is placed inside K^i. Then the planar subdrawing of K^i induced by D divides the plane as shown in Fig. 2(a) and, either $\mathrm{cr}_D(K^i, K^{i-1}) \geq 2$ or at least one edge of M_K^i joining K^i to K^{i-1} crosses K^i. Moreover, both paths $d_{i-1}e_{i-1}e_ie_{i+1}d_{i+1}$ and $d_{i-1}d_{i-2}e_{i-2}f_{i-2}f_{i-1}f_if_{i+1}f_{i+2}e_{i+2}d_{i+2}d_{i+1}$ cross the subgraph K^i in D. So, this contradicts the assumption that every subgraph G_1^i has at most two crossings on its edges. Thus, none of the subgraphs K^{i-1}, K^i and K^{i+1} separates two other and by Lemma 2, they do not cross each other. By Lemma 3, if K^i has internal crossing, every subdrawing $D(Q^i)$, $i = 2, 3, \ldots, n-2$, contains at least three crossings, every of types (1), (2) or (3).

Now we show that, in D, no edge of M_K^i crosses K^{i+1} as well as no edge of M_K^{i+1} crosses K^{i-1}, if K^i has a planar drawing. Without loss of generality, let an edge of M_K^{i+1} crosses K^{i-1}. As two different K^i and K^j do not cross and every of the subgraphs G_1^i has at most two crossings on its edges, $\mathrm{cr}_D(K^{i-1}, M_K^{i+1}) = 2$ and $f(Q^i) \geq 2$. Then the subdrawing of D induced by the subgraph $K^i \cup M_K^{i+1} \cup K^{i+1}$ without crossings divides the plane in such a way that at most two vertices of K^i are on the boundary of every region outside K^i. Therefore, at least one vertex of K^i is not on the boundaries of the regions with the vertices of K^{i-1} inside. This requires at least one crossing between an edge of M_K^i and an edge of $K^i \cup M_K^{i+1} \cup K^{i+1}$ and so, $f(Q^i) \geq 3$. So, $D(K^{i-1} \cup M_K^i \cup K^i)$ divides the plane in such a way that on the boundary of every region outside K^i there are at most two vertices of K^i, see the drawings in Fig. 3(a), where possible crossings among the edges of $K^{i-1} \cup M_K^i$ are inside the dotted cycle. As $\mathrm{cr}_D(K^{i+1}, K^{i-1} \cup M_K^i \cup K^i) = 0$, it is easy to verify that every placing of the subgraph K^{i+1} outside K^i enforces at least two crossings between the edges of M_K^{i+1} and the edges of $K^{i-1} \cup M_K^i \cup K^i$. Thus, $f(Q^i) \geq 2$. So, the unique drawing of the subgraph $K^{i-1} \cup M_K^i \cup K^i$ up to labeling the vertices is shown in Fig. 3(a). The vertices e_i and f_i are placed outside the subgraph K_i. Otherwise at least one of the edges e_ie_{i-1} and f_if_{i-1} crosses K^i and so, $f(Q^i) \geq 3$. Moreover, the subgraph K^{i+1} is placed inside the region where is situated the vertex e_i. Otherwise the edge e_ie_{i+1} crosses the subgraph $K^{i-1} \cup M_K^i \cup K^i$. Thus $f(Q^i) \geq 3$ and we are done. Then there is one

of the following cases: the cycle $d_ie_ie_{i-1}d_{i-1}d_i$ crosses the subgraph K^{i+1}, or the subgraph K^{i+1} is placed inside or outside this cycle. In the first case, the cycle $d_ie_ie_{i-1}d_{i-1}d_i$ crosses the subgraph K^{i+1} at least twice. These crossings are of type (1) or (4) and so, $f(Q^i) \geq 3$. The subgraph K^{i+1} can not be placed inside the cycle $d_ie_ie_{i-1}d_{i-1}d_i$, because the remaining three vertices of K^i are placed outside this cycle. It enforces at least three crossings between the edges of M_K^{i+1} and the edges of $K^{i-1} \cup M_K^i \cup K^i$ and so, $f(Q^i) \geq 3$ again. This implies that the subgraph K^{i+1} is placed outside the cycle $d_ie_ie_{i-1}d_{i-1}d_i$. It is easy to see that two edge-disjoint cycles separate one of the vertices a_i or c_i and the subgraph K^{i+1}. Without loss of generality, let it be the vertex c_i and so, the vertex c_i is placed inside the edge-disjoint cycles $d_ie_ie_{i-1}d_{i-1}a_{i-1}a_id_i$ and $d_id_{i-1}b_{i-1}b_id_i$. Moreover, as $\mathrm{cr}_D(K^{i+1}, K^{i-1} \cup M_K^i \cup K^i) = 0$ and the subgraph K^{i+1} is placed inside the region where is situated the vertex e_i, the cycle $d_ie_ie_{i-1}d_{i-1}a_{i-1}a_id_i$ separates the vertex b_i and the subgraph K^{i+1}. So, there are at least three crossings of the edges in subgraph M_K^{i+1} with the edges in $K^{i-1} \cup M_K^i \cup K^i \cup d_ie_ie_{i-1}d_{i-1}$, which every of them contribute the value 1 to $f(Q^i)$ and we are done. The similar way we can prove that if two edge-disjoint cycles separate one of the vertices a_i and the subgraph K^{i+1}, then also $f(Q^i) \geq 3$.

It remains to prove that $f(Q^i) \geq 2$ for $i = 1$ and $i = n-1$. Consider first the subgraph Q^1 and let $f(Q^1) < 2$. By Lemma 1, K^0 does not separate K^1 and K^2, K^2 does not separate K^1 and K^0, and if K^1 has an internal crossing, K^1 does not separate K^0 and K^2. So, it remains to prove that K^1 does not separate K^0 and K^2, if K^1 has a planar drawing. Otherwise, the planar subdrawing of K^1 induced by D divides the plane as shown in Fig. 2(a) and, either $\mathrm{cr}_D(K^0, K^1) \geq 2$ or at least one edge of M_K^1 joining K^0 to K^1 crosses K^1. So, $f(Q^1) \geq 1$. Moreover, the path $d_0e_0e_1e_2d_2$ crosses the subgraph K^1 and so, $f(Q^1) \geq \frac{3}{2}$. The force $f(Q^1)$ of Q^1 is equal to $\frac{3}{2}$, if the edge e_2d_2 crosses the subgraph K^1. As it was proved above that the subgraph K^1 can not separate K^3, the edge e_2e_3 crosses the subgraph K^1. This contradicts the assumption that every subgraph G_1^i has at most two crossings on its edges. It implies that $f(Q^1) \geq 2$. So, none of the subgraphs K^0, K^1 and K^2 separates two other. Moreover, by Lemma 2, the subgraphs K^0, K^1 and K^2 do not cross each other. This implies that no edge of M_K^1 crosses K^2 as well as no edge of M_K^2 crosses K^0. Otherwise $\mathrm{cr}_D(K^0, M_K^2) = \mathrm{cr}_D(K^2, M_K^1) = 2$ and $f(Q^i) \geq 2$. It implies, that regardless of the edges of K^1 cross each other or not, $D(K^0 \cup M_K^1 \cup K^1)$ divides the plane in such a way that on the boundary of every region outside K^1 there are at most two vertices of K^1, see the drawings in Fig. 3, where possible crossings among the edges of $K^0 \cup M_K^1$ are inside the dotted cycle. As $\mathrm{cr}_D(K^2, K^0 \cup M_K^1 \cup K^1) = 0$, it is easy to verify that every placing of the subgraph K^2 outside K^1 enforces at least two crossings between the edges of M_K^2 and the edges of $K^0 \cup M_K^1 \cup K^1$. Thus, $f(Q^1) \geq 2$ and this is contradiction with assumption $f(Q^1) < 2$. The similar analysis for the subdrawing of Q^{n-1} gives $f(Q^{n-1}) \geq 2$.

So, this enforces that, there are at least $\sum_{i=1}^{n-1} \mathrm{cr}_D(Q^i) = 4 + 3(n-3) = 3n-5$ crossings among the edges of the subgraph $G_1^0 \cup M^1 \cup G_1^1 \cup \cdots \cup G_1^{n-1} \cup M^n \cup G_1^n$,

in D. This completes the proof. □

For the crossing number of the graph $G_1 \square P_n$ for $n \geq 3$ we have the next result.

Theorem 1 $\mathrm{cr}(G_1 \square P_n) = 3n - 5$ *for* $n \geq 3$.

Proof. The drawing in Fig. 2(c) with $3n - 5$ crossings confirms that $\mathrm{cr}(G_1 \square P_n) \leq 3n - 5$ for $n \geq 3$. We prove the reverse inequality by induction on n. The graph $S_4 \square P_3$ is a subgraph of $G_1 \square P_3$ and we know that $\mathrm{cr}(S_4 \square P_3) = 4$ (see [1]). Thus, the crossing number of $G_1 \square P_3$ is at least four and the result is true for $n = 3$. Assume that it is true for $n = k$, $k \geq 3$, and suppose that there is a good drawing of the graph $G_1 \square P_{k+1}$ with fewer than $3k - 2$ crossings. By Lemma 4, some of the subgraphs G_1^i, $i = 0, 1, ..., k + 1$, must be crossed at least three times. If G_1^0 has at least three crossings on its edges, then deleting of all vertices of G_1^0 results in a drawing of the graph $G_1 \square P_k$ with fewer than $3k-5$ crossings. This contradicts the induction hypothesis. The same contradiction is obtained if at least three crossings appear on the edges of G_1^{k+1}. If some G_1^i, $i \in \{1, 2, ..., k\}$, is crossed at least three times, by the removal of all edges of this G_1^i, a subdivision of $G_1 \square P_k$ with fewer than $3k-5$ crossings is obtained. This contradiction with the induction hypothesis completes the proof. □

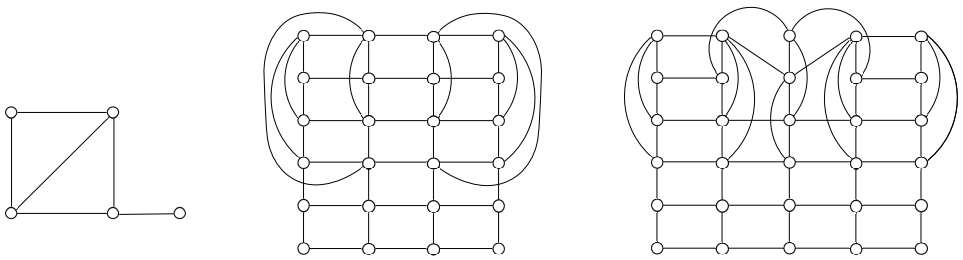

Figure 4: The graphs H, $G_2 \square P_3$ and $G_2 \square P_4$

For the crossing number of the graph $G_2 \square P_n$ for $n \geq 1$ we have the next results.

Theorem 2 $\mathrm{cr}(G_2 \square P_n) = 2(n - 1)$ *for* $n \geq 1$.

Proof. It is easy to see that the graph $G_2 \square P_1$ is planar. In Fig. 4(b), Fig. 4(c) and Fig. 5 there are the drawings of the graph $G_2 \square P_n$ for $n \geq 2$ with $2(n - 1)$ crossings. The reverse inequality follows from the fact that the graph H presented in Fig. 4(a) is the subgraph of G_2 and $\mathrm{cr}(H \square P_n) = 2(n - 1)$ (see [10]). □

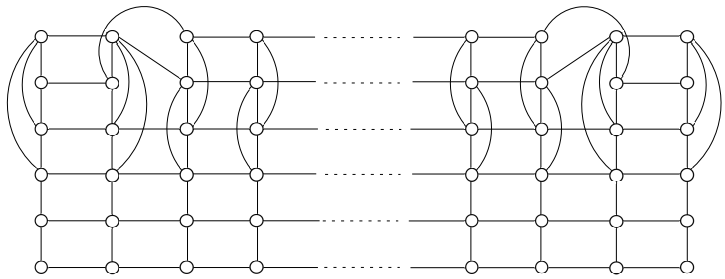

Figure 5: The graph $G_2 \square P_n$ for $n \geq 5$

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Theory of SSB Representation of Preferences Revised*

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Abstract

A continuous skew-symmetric bilinear (SSB) representation of preferences has recently been proposed in a topological vector space, assuming a weaker notion of convexity of preferences than in the classical (algebraic) case. Equipping a linear vector space with the so-called inductive linear topology, we derive the algebraic SSB representation on a topological basis, thus weakening the convexity assumption. Such a unifying approach to SSB representation permits also to fully discuss the relationship of topological and algebraic axioms of continuity, and leads to a stronger existence result for a maximal element. By applying this theory to probability measures we show the existence of a maximal preferred measure for an infinite set of pure outcomes, thus generalizing all available existence theorems in this context.

1 Introduction

Many systematic violations of the *expected utility theory* [13] have been observed, see e.g. [12], stimulating the development of alternative decision-making theories [5, 11, 8]. In particular, the axiom of *transitivity* of preferences, nowadays understood as an intuitively appealing cornerstone of rationality, is not always supported by empirical evidence [2]. A concise mathematical model of non-transitive decision-making has been proposed in [3], representing preferences with a skew-symmetric bilinear (SSB) functional. Note that from the mathematical point of view, such representation is closely related to the regret theory [7], see [1].

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Denoting $\succ$ an asymmetric relation of strict preferences on a non-empty convex set P, we say that a functional ϕ on $P \times P$ is an SSB representation of $\succ$ if ϕ is SSB and $p \succ q \Rightarrow \phi(p,q) > 0$ for all $p, q \in P$. Let $\sim$ and $\succsim$ be indifference and preference-or-indifference relations defined in a standard way using $\succ$. Then, the axioms of (algebraic) SSB representation stated for all $p, q, r \in P$ and all $\lambda \in]0,1[$ are as follows:

$$
\begin{array}{lll}
\text{(C1)} & \text{Continuity:} & p \succ q, q \succ r \Longrightarrow q \sim \alpha p + (1-\alpha) r \text{ for some } \alpha \in]0,1[, \\
\text{(C2)} & \text{Convexity:} & p \succ q, p \succsim r \Longrightarrow p \succ \lambda q + (1-\lambda) r, \\
 & & p \sim q, p \sim r \Longrightarrow p \sim \lambda q + (1-\lambda) r, \\
 & & q \succ p, r \succsim p \Longrightarrow \lambda q + (1-\lambda) r \succ p, \\
\text{(C3)} & \text{Symmetry}^1\text{:} & p \succ q, q \succ r, p \succ r \Longrightarrow \Big[q \sim \dfrac{p+r}{2} \Longrightarrow \\
 & & \Big(\lambda p + (1-\lambda) r \sim \dfrac{p+q}{2} \Rightarrow \lambda r + (1-\lambda) p \sim \dfrac{r+q}{2} \Big) \Big].
\end{array}
$$

If P is, moreover, a set of probability measures, axioms (C1)–(C3) hold if and only if there exists an SSB representation of $\succ$, see [3, Theorem 1].

Recently, a variant of an SSB representation of preferences has been proposed in a topological vector space [9]. For a non-empty convex subset P (being equipped with the relative topology) the axioms for all $p, q, r \in P$ and $\lambda \in]0,1[$ are the following:

$$
\begin{array}{lll}
\text{(F1)} & \text{Continuity:} & \text{sets } \{s \in P : p \succ s\} \text{ and } \{s \in P : s \succ p\} \text{ are open}, \\
\text{(F2)} & \text{Convexity:} & p \succ q, p \succsim r \Longrightarrow p \succ \lambda q + (1-\lambda) r, \\
 & & q \succ p, r \succsim p \Longrightarrow \lambda q + (1-\lambda) r \succ p, \\
\text{(F3)} & \text{Balance}^1\text{:} & q \sim \dfrac{p+r}{2}, \lambda p + (1-\lambda) r \sim \dfrac{p+q}{2} \Longrightarrow \\
 & & \lambda r + (1-\lambda) p \sim \dfrac{r+q}{2}.
\end{array}
$$

An asymmetric binary relation $\succ$ on P satisfies (F1), (F2) and (F3) if and only if there exists an SSB representation of $\succ$ that is, moreover, separately continuous in each variable, see Theorem 1 below. Further, we have shown that in a compact and convex subset of P there exists a maximal element with respect to $\succ$, see Theorem 3 below. Consequently, we have generalized the existence result for a maximal element in the case of an infinite set of outcomes, see Theorem 5.

Finally, equipping a linear vector space X with the so-called *inductive linear topology* [6], i.e. the finest topology such that X is a Hausdorff t.v.s. and for any finite-dimensional subspace Y of X canonical injection of Y into X is continuous, the algebraic SSB representation may be considered as an application of the above introduced topological theory [10]. Such an observation leads to a generalization of

[1] We use a slightly adapted variant of axiom (C3) to stress that the conclusion of axiom (C3) is equivalent to axiom (F3). Note that axioms (C3) and (F3) are equivalent given axioms (C1) and (F2), see [10, Theorem 4.2].

the algebraic SSB representation theorem, as well as the theorem for the existence of a maximal element in a linear vector space, see Corollary 2 and Corollary 4, respectively. The proposed technique may be of general interest since it permits one to use topological tools to obtain relatively stronger results that may be finally transposed to a purely algebraic setting employing the inductive linear topology.

The basic notation used is standard. A *topological space* is a set X equipped with a family of subsets $\subset 2^X$ (called *open* sets) satisfying the following conditions: $\emptyset, X \in$; every union of open subsets of X is open; every finite intersection of open subsets of X is open. A topological space is *compact* if each of its open covers has a finite sub-cover; is a *Hausdorff* space if any two distinct points are respectively contained in disjoint open sets; is a real *topological vector space* (t.v.s) if it is moreover a real *linear vector space* (l.v.s.) such that operations of addition and multiplication are continuous. By $\mathscr{P}(X)$ we denote a set of all *regular Borel probability* measures on X equipped with the so-called weak* topology.

2 Main Results

First, we present the topological version of the SSB representation theorem, see [9, Theorem 3.6 and Theorem 5.3].

Theorem 1. *Let P be a non-empty convex subset of a t.v.s. equipped with the relative topology. An asymmetric relation on P satisfies (F1), (F2) and (F3) if and only if there exists a separately continuous SSB functional ϕ on $P \times P$ such that for all $p, q \in P$, p q $\phi(p, q) > 0$.*

Transposing the above theorem in a l.v.s. with the use of inductive linear topology, one obtains the following generalization of [3, Theorem 1], see [10, Theorem 4.2].

Corollary 2. *Let P be a non-empty convex subset of a l.v.s. A binary relation on P satisfies (C1), (F2) and (F3) if and only if there exists an SSB functional ϕ on $P \times P$ such that for all $p, q \in P$, p q $\phi(p, q) > 0$.*

Comparing the statements of Theorem 1 and Corollary 2, we see that axiom (C1) plays two different roles. First, it implies asymmetry of that has to be explicitly assumed in Theorem 1. Besides, it amounts to continuity axiom (F1) in the algebraic setting; indeed, any SSB functional is separately continuous with respect to inductive linear topology, see [6, Proposition 4.1.2 and Proposition 4.5.4].

Next, we show that standard continuity and convexity assumptions imply the existence of a maximal and a minimal element in a t.v.s., see [9, Corollary 3.4 and Theorem 3.6].

Theorem 3. *Let P be a non-empty compact convex subset of a Hausdorff t.v.s., and be an asymmetric relation on P satisfying axioms (F1) and (F2), then there exist a minimal and a maximal element of P with respect to .*

Employing the inductive linear topology again, see [10, Theorem 4.3], one obtains an analogous statement in a l.v.s.

Corollary 4. *Let P be a non-empty compact convex subset of a finite-dimensional l.v.s., and be a relation that satisfies (C1) and (F2) on P, then there exist a minimal and a maximal element of P with respect to .*

Note that previously a similar existence result that has been shown only for a (finitely generated) polyhedral subset of P assuming, moreover, axiom (C3), see [5, Theorem 6.2].

Finally, we generalize [4, Theorem 5] on the basis of Theorem 1 and Theorem 3, for a detailed proof see [9, Theorem 6.2].

Theorem 5. *Let set of outcomes X be a compact Hausdorff space and ϕ be a bounded real function on $X \times X$ that is separately continuous in each variable and satisfies $\phi(x, y) = -\phi(y, x)$ for all $x, y \in X$. Define functional on $\mathscr{P}(X) \times \mathscr{P}(X)$ by*

$$(p, q) \equiv \int_{X \times X} \phi(x, y) dp(x) dq(y).$$

Then is a separately continuous SSB functional on $\mathscr{P}(X)$, and for a closed and convex set $K \subset \mathscr{P}(X)$ there exists $p \in K$ such that $(p, q) \geq 0$ for all $q \in K$.

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New Approach How to Generate Priority Vector to Pairwise Comparisons Matrix With Fuzzy Elements

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Abstract. In this paper we deal with Condition of Order Preservation (COP) of pairwise comparisons (PC) matrix with fuzzy elements. Fuzzy elements are appropriate whenever the decision maker (DM) is uncertain about the value of his/her evaluation of the relative importance of elements in question, or, when aggregating crisp pairwise comparisons of a group of decision makers in the group DM problem. We formulate the problem in a general setting investigating pairwise comparisons matrices with elements from abelian linearly ordered group (alo-group). Such an approach enables extensions of traditional multiplicative, additive or fuzzy approaches. We review the approaches known from the literature, then we propose our new order preservation concept based on alpha-cuts. We define the concept of consistency of PC matrix with fuzzy elements (FPC matrices). We derive the necessary and sufficient conditions for strict consistency as well as weak and strong POP conditions and relationships. Finally, we deal with some consequences to the problem of ranking the alternatives. Illustrating examples are presented and discussed.

1 Introduction

The problem we consider here is as follows: Let $\mathcal{C} = \{c_1, c_2, ..., c_n\}$ be a finite set of alternatives ($n > 1$). The goal of the DM is to rank the alternatives from the best to the worst (or, vice versa, which is equivalent), using the information given by the decision maker in the form of an $n \times n$ pairwise comparisons matrix (PCM). The ranking of the alternatives is determined by the priority vector of real numbers $w = (w_1, w_2, ..., w_n)$ which is calculated from the corresponding PCM. There exist various methods for calculating the vector of weights based on the DM problem, particularly, on the pairwise comparisons matrix, see e.g. [7].

Fuzzy sets as the elements of the pairwise comparisons matrix can be applied in the DM problem whenever the decision maker is not sure about the preference degree of his/her evaluations of the pairs in question. Fuzzy elements are useful in order to capture uncertainty stemming from subjectivity of human thinking and from incompleteness of information that is an integral part of multi-criteria

decision-making problems. Fuzzy elements may be also useful as aggregations of crisp pairwise comparisons of a group of decision makers in the group DM problem.

In [10] and [11], the author presented a general approach for PCM with fuzzy number elements based on alo-groups unifying the previous approaches. In [12], the concept of strong consistency was introduced and some relationships with the (ordinary) consistency was derived. In comparison to [10] and [11]. Fuzzy intervals are elements of the PCM called Fuzzy Pairwise Comparisons matrix (FPCM).

In this paper we follow definitions from [12], generalize the concept of COP to FPC matrices defining α-Weak COP and α-Strong COP. Here, we reconsider Generalized Geometric Mean Method (GGMM) and show that satisfying "fuzzy" COP, the criteria under a generalized GMM depend on the locally defined inconsistency. Furthermore, we solve the problem of measuring inconsistency of FPC matrices by defining a corresponding index. Finally, we discuss several numerical examples in order to illustrate the proposed concepts and properties.

2 Preliminaries

The reader can find the corresponding basic definitions, concepts and results in the previous proceedings of the CJS seminar, [12]. Remember, that for X, a nonempty subset of the n-dimensional Euclidean space $\mathbf{R}^n$, a fuzzy set $S = (a, b, c, d)$ in X is called *closed, bounded, compact* or *convex* if the α-cut $[S]_\alpha$ is a closed, bounded, compact or convex subset of X for every $\alpha \in]0; 1]$, respectively.

In order to unify various approaches and prepare a more flexible presentation, we apply alo-groups, see [12]. Recall that an abelian group, [2], is a set, G, together with an operation $\odot$ and corresponding group axioms that combines any two elements $a, b \in G$ to form another element in G denoted by $a \odot b$, see [2]. The well known examples of alo-groups can be found in [12], or, [11].

Example 1. Additive alo-group $\mathcal{R} = (\mathbf{R}, +, \leq)$ is a continuous alo-group with: $e = 0,\ a^{(-1)} = -a$.

Example 2. Multiplicative alo-group $\mathcal{R}_+ = (\mathbf{R}_+, \bullet, \leq)$ is a continuous alo-group with: $e = 1,\ a^{(-1)} = a^{-1} = 1/a$. Here, by $\bullet$ we denote the usual operation of multiplication.

Example 3. Fuzzy additive alo-group $\mathcal{R}_a$=$(\mathbf{R}, +_f, \leq)$, see [12], is a continuous alo-group with: $a +_f b = a + b - 0.5,\ e = 0.5,\ a^{(-1)} = 1 - a$.

Example 4. Fuzzy multiplicative alo-group $\mathcal{R}_m$=$(]0; 1[, \bullet_f, \leq)$, see [5], is a continuous alo-group with: $a \bullet_f b = \frac{ab}{ab+(1-a)(1-b)}, e = 0.5, a^{(-1)} = 1 - a$.

3 FPC matrices, reciprocity and consistency

Our general approach based on alo-groups is useful, as it unifies various important approaches known from the literature. This fact has been already demonstrated on 4

examples presented above, where the well known alo-groups are shown. Particularly, all concepts and properties which will be presented bellow can be easily applied to any alo-group. Before we shall investigate PC matrices with fuzzy elements we remember some concepts and properties of PC matrices on alo-group with crisp elements.

A crisp PC matrix $A = \{a_{ij}\}$ is said to be $\odot$-*reciprocal*, if the following condition holds: For every $i, j \in \{1, ..., n\}$

$$a_{ij} \odot a_{ji} = e, \text{ or, equivalently, } a_{ji} = a_{ij}^{(-1)}. \tag{1}$$

A crisp FPC matrix $A = \{a_{ij}\}$ is $\odot$-*consistent* if for all $i, j, k \in \{1, ..., n\}$

$$a_{ik} = a_{ij} \odot a_{jk}, \text{ or, equivalently, } a_{ij} \odot a_{jk} \odot a_{ki} = e. \tag{2}$$

Remember that an $\odot$-consistent PC matrix $A = \{a_{ij}\}$ is $\odot$-reciprocal, but not vice-versa. The following equivalent condition for consistency of PC matrices is well known, see e.g. [5], [14].

A crisp PC matrix $A = \{a_{ij}\}$ is $\odot$-consistent if and only if there exists a vector $w = (w_1, ..., w_n)$, $w_i \in G$, such that

$$a_{ij} = w_i \div w_j \text{ for all } i, j \in \{1, 2, ..., n\}. \tag{3}$$

Here, $w_i \div w_j = w_i \odot w_j^{(-1)}$.

In [12], we extended the above stated definition of $\odot$-reciprocity and $\odot$-consistency to non-crisp matrices with fuzzy elements. Particularly, we introduce a new concept of reciprocity and consistency based on α-cuts: α-$\odot$-reciprocity and α-$\odot$-consistency. We start, however, with the α-$\odot$-reciprocity in the crisp case.

Let $\mathcal{G} = (G, \odot, \leq)$ be a divisible and continuous alo-group over an open interval G of $\mathbf{R}$. Let $\alpha \in [0; 1]$, $\tilde{A} = \{\tilde{a}_{ij}\}$ be an $n \times n$ matrix, where each element is a bounded fuzzy interval of the alo-group $\mathcal{G}$, let $[\tilde{a}_{ij}]_\alpha = [a_{ij}^L(\alpha), a_{ij}^R(\alpha)]$ be an α-cut of $\tilde{a}_{ij}$.

Matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is said to be α-$\odot$-*reciprocal*, if the following two conditions hold for each $i, j \in \{1, ..., n\}$:

$$a_{ii}^L(\alpha) = a_{ii}^R(\alpha) = e, \tag{4}$$

$$a_{ij}^L(\alpha) \odot a_{ji}^R(\alpha) = e. \tag{5}$$

If $\tilde{A} = \{\tilde{a}_{ij}\}$ is α-$\odot$-reciprocal for all $\alpha \in [0; 1]$, then it is called $\odot$-*reciprocal*.

If $\tilde{A} = \{\tilde{a}_{ij}\}$ is $\odot$-reciprocal, then $\tilde{A} = \{\tilde{a}_{ij}\}$ is called the *fuzzy pairwise comparisons matrix*, *fuzzy PC matrix, FPC matrix*, or, shortly, *FPCM*.

Now, we turn to the concept of consistency of FPC matrices. We start with the definition of *weak* α-$\odot$-consistent FPC matrix. Later on, we shall define a stronger concept, particularly, an α-$\odot$-consistency of FPC matrix.

Definition 1. *Let* $\alpha \in [0; 1]$. *A FPC matrix* $\tilde{A} = \{\tilde{a}_{ij}\}$ *is said to be* weak α-$\odot$-consistent*, if the following condition holds:*

There exists a crisp matrix $A' = \{a'_{ij}\}$ *with* $a'_{ik} \in [\tilde{a}_{ik}]_\alpha$, $a'_{ij} \in [\tilde{a}_{ij}]_\alpha$, $a'_{jk} \in [\tilde{a}_{jk}]_\alpha$, *such that* $A' = \{a'_{ij}\}$ *is consistent, i.e. for each* $i, j, k \in \{1, ..., n\}$ *it holds*

$$a'_{ik} = a'_{ij} \odot a'_{jk}. \tag{6}$$

The FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ *is said to be* weak $\odot$-consistent, *if* $\tilde{A}$ *is weak* α-$\odot$-*consistent for all* $\alpha \in [0; 1]$.
If for some $\alpha \in [0; 1]$ *the FPC matrix* $\tilde{A} = \{\tilde{a}_{ij}\}$ *is not weak* α-$\odot$-*consistent, then* $\tilde{A}$ *is called* α-$\odot$-inconsistent.

Ramarks. Let $\alpha, \beta \in [0; 1], \alpha \geq \beta$.

- For a crisp PCM, definitions of $\odot$-reciprocity and $\odot$-consistency in Definition 1 coincide with the classical definitions.
- If $\tilde{A} = \{\tilde{a}_{ij}\}$ is weak α-$\odot$-consistent, then it is weak β-$\odot$-consistent.
- (5) holds for all $i, j \in \{1, ..., n\}$ if and only if (5) holds for all $i, j \in \{1, ..., n\}$, $1 \leq i < j \leq n$.
- (6) holds for all $i, j, k \in \{1, ..., n\}$ if and only if (6) holds for all $i, j, k \in \{1, ..., n\}, 1 \leq i < j < k \leq n$.

The next proposition gives four equivalent conditions for a FPC matrix to be weak α-$\odot$-consistent, for the proof see [12].

Proposition 1. *Let* $\alpha \in [0; 1]$, *let* $\tilde{A} = \{\tilde{a}_{ij}\}$ *be a FPC matrix,* $[\tilde{a}_{ij}]_\alpha = [a^L_{ij}(\alpha), a^R_{ij}(\alpha)]$ *be an* α*-cut of* $\tilde{a}_{ij}$. *The following conditions are equivalent.*

- $\tilde{A} = \{\tilde{a}_{ij}\}$ *is weak* α-$\odot$-*consistent.*
- *There exists a vector* $w = (w_1, ..., w_n)$ *with* $w_i \in G, i \in \{1, ..., n\}$, *such that for each* $i, k \in \{1, ..., n\}$, *it holds:*

$$a^L_{ik}(\alpha) \leq w_i \div w_k \leq a^R_{ik}(\alpha). \tag{7}$$

- *For each* $i, j, k \in \{1, ..., n\}$, *it holds:*

$$a^L_{ik}(\alpha) \leq a^R_{ij}(\alpha) \odot a^R_{jk}(\alpha), \tag{8}$$

- *For each* $i, j, k \in \{1, ..., n\}$, *it holds:*

$$a^R_{ik}(\alpha) \geq a^L_{ij}(\alpha) \odot a^L_{jk}(\alpha). \tag{9}$$

Example 5. Consider the additive alo-group $\mathcal{R} = (\mathbf{R}, \odot, \leq)$ with $\odot = +$, see Example 1. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be given by triangular fuzzy number elements as follows:

$$\tilde{A} = \begin{bmatrix} (0,0,0) & (1,3,4) & (4,6,8) \\ (-4,-3,-1) & (0,0,0) & (2,4,5) \\ (-8,-6,-4) & (-5,-4,-2) & (0,0,0) \end{bmatrix},$$

or, equivalently, by α-cut notation, each fuzzy set is given by the corresponding family of α-cuts, i.e. intervals. Particularly, for $\alpha \in [0;1]$, we obtain:

$$\tilde{A} = \begin{bmatrix} [0;0] & [1+2\alpha; 4-\alpha] & [4+2\alpha; 8-2\alpha] \\ [-4+\alpha; -1-2\alpha] & [0;0] & [2+2\alpha; 5-\alpha] \\ [-8+2\alpha; -4-2\alpha] & [-5+\alpha; -2-2\alpha] & [0;0] \end{bmatrix}.$$

Here, $\tilde{A}$ is a 3×3 matrix with triangular fuzzy number elements and the corresponding piece-wise linear membership functions. Conditions (4) and (5) can be easily verified for all $\alpha \in [0;1]$, hence, $\tilde{A}$ is $\odot$-reciprocal.
By Proposition 1 we check only one of inequalities (8) (or (9)), for all triples of indices $i, j, k \in \{1, 2, 3\}, i < j < k$. By simple calculations, we obtain that $\tilde{A}$ is weak α-$\odot$-consistent for all $0 \leq \alpha \leq \frac{5}{6}$.
It is also evident that $\tilde{A}$ is α-$\odot$-inconsistent FPCM for $\frac{5}{6} < \alpha \leq 1$.

Now, we are going to define a stronger concept of α-$\odot$-consistency (without adjective "weak") based on formula similar to (2) in the crisp case. In the literature there exist more other concepts of consistency, e.g. Liu's consistency, approximate consistency, or strong consistency, see e.g. [13]. These approaches are not, however, investigated here.

Definition 2. *Let $\alpha \in [0;1]$. A FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is said to be* α-$\odot$-consistent, *if the following condition holds:*
For all $i, j, k \in \{1, ..., n\}$, it holds:

$$a_{ij}^L(\alpha) \odot a_{jk}^L(\alpha) \odot a_{ki}^L(\alpha) = a_{ik}^L(\alpha) \odot a_{kj}^L(\alpha) \odot a_{ji}^L(\alpha), \tag{10}$$

$$a_{ij}^R(\alpha) \odot a_{jk}^R(\alpha) \odot a_{ki}^R(\alpha) = a_{ik}^R(\alpha) \odot a_{kj}^R(\alpha) \odot a_{ji}^R(\alpha). \tag{11}$$

Moreover, if $\tilde{A} = \{\tilde{a}_{ij}\}$ is α-$\odot$-consistent for all $\alpha \in [0;1]$, then $\tilde{A}$ is said to be $\odot$-consistent.

An α-$\odot$-consistent FPC matrix is not necessarily α-$\odot$-reciprocal, as it is in the crisp case. In real DM problems, α-$\odot$-reciprocity condition is, however, a natural assumption. Therefore, in the sequel we assume that FPCMs are always -$\odot$-reciprocal. The following proposition gives a characterization of α-$\odot$-reciprocal α-$\odot$-consistent FPC matrices, see also [6]. The proof is straightforward and is left to the reader.

Proposition 2. *Let $\alpha \in [0;1]$, let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a reciprocal FPC matrix, $[\tilde{a}_{ij}]_\alpha = [a_{ij}^L(\alpha), a_{ij}^R(\alpha)]$ be an α-cut.*
$\tilde{A} = \{\tilde{a}_{ij}\}$ is α-$\odot$-consistent if and only if for all $i, j, k \in \{1, 2, ..., n\}$, it holds:

$$a_{ik}^L(\alpha) \odot a_{ik}^R(\alpha) = a_{ij}^L(\alpha) \odot a_{ij}^R(\alpha) \odot a_{jk}^L(\alpha) \odot a_{jk}^R(\alpha). \tag{12}$$

Definition 3. *Let $\alpha \in [0;1]$, let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix, $[\tilde{a}_{ij}]_\alpha = [a_{ij}^L(\alpha), a_{ij}^R(\alpha)]$ be an α-cut. For all $i, j \in \{1, ..., n\}$ denote*

$$a_{ij}^m(\alpha) = (a_{ij}^L(\alpha) \odot a_{ij}^R(\alpha))^{(\frac{1}{2})}. \tag{13}$$

A crisp $n \times n$ - matrix $A^m(\alpha) = \{a_{ij}^m(\alpha)\}$ is called α-$\odot$-mean matrix *associated to FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$.*

By (13) in Proposition 2, FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is α-$\odot$-consistent, if and only if crisp α-$\odot$-mean matrix $A^m(\alpha)$ is $\odot$-consistent, i.e. the following formula holds for all $i, j, k \in \{1, ..., n\}$:

$$a^m_{ij}(\alpha) = a^m_{ij}(\alpha) \odot a^m_{jk}(\alpha). \tag{14}$$

Moreover, if FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is α-$\odot$-consistent for all $\alpha \in [0; 1]$, then $\tilde{A}$ is $\odot$-consistent.

Remark 1. Notice that α-$\odot$-consistency of FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is equivalent to $\odot$-consistency of the associated crisp matrix, particularly, α-$\odot$-mean matrix associated to $\tilde{A}$, with elements being given by (13). This property will be advantageous in deriving a corresponding priority vector of the FPC matrix as we can see in the next section.

Example 6. Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be given FPC matrix from Example 5.
Moreover, α-+-mean matrix $A^m(\alpha)$ associated to $\tilde{A}$ for $\alpha \in [0, 1]$ is calculated by (13) as

$$A^m(\alpha) = \begin{bmatrix} 0 & \frac{5+\alpha}{2} & 6 \\ -\frac{5+\alpha}{2} & 0 & \frac{7+\alpha}{2} \\ -6 & -\frac{7+\alpha}{2} & 0 \end{bmatrix}.$$

Checking equality (14), we obtain that $\tilde{A}$ is α-+-consistent for all $\alpha = 0$.

The following propositions give some characterizations of α-$\odot$-consistent FPC matrices, see also [6]. The proofs are left to the reader.

Proposition 3. *Let $\alpha \in [0, 1]$, let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix, $[\tilde{a}_{ij}]_\alpha = [a^L_{ij}(\alpha), a^R_{ij}(\alpha)]$ be an α-cut.*
$\tilde{A} = \{\tilde{a}_{ij}\}$ is α-$\odot$-consistent if and only if there exists a vector $w(\alpha) = (w_1(\alpha), ..., w_n(\alpha))$ with $w_j(\alpha) \in G, j \in \{1, ..., n\}$, such that for each $i, k \in \{1, ..., n\}$, it holds:

$$a^m_{ik}(\alpha) = w_i(\alpha) \div w_k(\alpha). \tag{15}$$

Proposition 4. *Let $\alpha \in [0; 1]$, let $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix.*
If $\tilde{A}$ is α-$\odot$-consistent then $\tilde{A}$ is weak α-$\odot$-consistent. Moreover, if $\tilde{A}$ is $\odot$-consistent then $\tilde{A}$ is weak $\odot$-consistent.

4 Desirable properties of the priority vector

Pairwise comparisons matrices may violate some desirable conditions of multiple criteria decision making: e.g. the 'best' alternative with respect to DMs preferences is selected from the set of non-dominated alternatives, on condition this set is non-empty. The other PCMs may violate the conditions of order of preferences (the so called COP conditions), see Bana e Costa and Vansnick [1].

A PC matrix with crisp elements $A = \{a_{ij}\}$ is said to satisfy preservation of order preference condition (POP condition) with respect to priority vector w if

$$a_{ij} > e \Rightarrow w_i > w_j. \tag{16}$$

Finally, a PC matrix A is said to satisfy reliable preference (RP) condition with respect to priority vector w if

$$a_{ij} > e \Rightarrow w_i > w_j, \tag{17}$$

$$a_{ij} = e \Rightarrow w_i = w_j. \tag{18}$$

From (17) in the above definition it is evident that if a crisp PC matrix A satisfies RP condition with respect to priority vector w, then A satisfies POP condition with respect to priority vector w. The opposite is not true.

Let $A = \{a_{ij}\}$ be a crisp consistent PC matrix, and let $w = (w_1, ..., w_n)$ be a priority vector associated to A satisfying (3). Then it is obvious that FS, POP and RP conditions are satisfied. Moreover, it is well known, see e.g. [14], that for each crisp consistent PC matrix, the priority vector satisfying (3) can be generated either by the eigenvalue method (EVM), or, by the geometric mean method (GMM).

Now, we are going to extend the POP condition for a FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$, as we mentioned before.

Definition 4. *Let $c_i, c_j \in \mathcal{C}$, $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on alo-group $\mathcal{G} = (G, \odot, \leq), \alpha \in [0; 1]$.*

$$a_{ij}^m(\alpha) = (a_{ij}^L(\alpha) \odot a_{ij}^R(\alpha))^{(\frac{1}{2})}. \tag{19}$$

We say that c_i α-mean dominates *c_j, if $a_{ij}^m(\alpha) > e$.*
Moreover, c_i mean dominates *c_j, if c_i α-mean dominates c_j, for all $\alpha \in [0; 1]$.*

Definition 5. *Let $c_i, c_j \in \mathcal{C}$, $\tilde{A} = \{\tilde{a}_{ij}\}$ be a FPC matrix on alo-group $\mathcal{G} = (G, \odot, \leq)$, $w = (w_1, ..., w_n)$, $w_i \in G$, be a priority vector, $\alpha \in [0; 1]$.*
We say that the α-mean preservation of order preference condition (α-MPOP condition) *is satisfied:*
if c_i α-mean dominates c_j, then $w_i > w_j$.
We say that the mean preservation of order preference condition (MPOP condition) *is satisfied:*
if c_i mean dominates c_j, then $w_i > w_j$.

In what follows we introduce the local error indexes and global error index based on the $\odot$-mean of the end points of α-cuts of fuzzy elements of FPCM.

Let $x \in G$, the *norm of x in alo-group G*, $\|x\|$, is defined as

$$\|x\| = \max\{x, x^{(-1)}\}. \tag{20}$$

Definition 6. *Let $\tilde{A} = \{\tilde{a}_{ij}\}$ be an FPC matrix on alo-group $\mathcal{G} = (G, \odot, \leq)$. For each pair $i, j \in \{1, \ldots, n\}$, and a priority vector $w = (w_1, ..., w_n)$, $w_i \in G$, $\alpha \in [0; 1]$ and for $a_{ij}^m(\alpha) = (a_{ij}^L(\alpha) \odot a_{ij}^R(\alpha))^{(\frac{1}{2})}$ let us denote:*

$$\varepsilon^m(i, j, w, \alpha) = \|a_{ij}^m(\alpha) \odot w_j \div w_i\|, \tag{21}$$

The global error index, *$\mathcal{E}(\tilde{A}, w, \alpha)$, for a FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ and a priority vector $w = (w_1, \ldots, w_n)$, is defined as the maximal value of $\varepsilon^m(i, j, w, \alpha)$, i.e.*

$$\mathcal{E}(\tilde{A}, w, \alpha) = \max_{i,j \in \{1,\ldots,n\}} \varepsilon^m(i, j, w, \alpha). \tag{22}$$

5 Priority vectors, measuring inconsistency of FPC matrices

In this section we extend our considerations from crisp PC matrix $A = \{a_{ij}\}$ to FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ by using crisp α-$\odot$-mean matrix $A^m(\alpha) = \{a_{ij}^m(\alpha)\}$ associated to FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$. Here, the elements of crisp PC matrix $A^m(\alpha)$ are defined by (13) as

$$a_{ij}^m(\alpha) = (a_{ij}^L(\alpha) \odot a_{ij}^R(\alpha))^{(\frac{1}{2})},$$

depending on the given $\alpha \in [0; 1]$.

We propose a method for calculating the priority vector of $n \times n$ FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ for the purpose of rating the alternatives $c_1, ..., c_n \in \mathcal{C}$. Here, we do not follow the way of calculating the *fuzzy* priority vector proposed e.g. in [4], and others. Here, we generate a crisp priority vector, therefore, no defuzzification will be necessary for final ranking the alternatives. The proposed method for calculating the priority vector can be divided into two steps as follows.

Step 1.
In *Step 1* we check whether the given FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is weak α-$\odot$-consistent for some α, where $0 \leq \alpha \leq 1$. Then we calculate the maximal such α denoted by α^*. By Remark 3, FPCM $\tilde{A}$ is therefore weak α-$\odot$-consistent for all $\alpha \leq \alpha^*$. The following optimization problem is solved:

(P1)

$$\alpha \longrightarrow \max; \tag{23}$$

subject to

$$a_{ij}^L(\alpha) \leq w_i \div w_j \leq a_{ij}^R(\alpha) \text{ for all } i, j \in \{1, ..., n\}, \tag{24}$$

$$\bigodot_{k=1}^{n} w_k = e, \tag{25}$$

$$0 \leq \alpha \leq 1, w_k \in G, \text{ for all } k \in \{1, ..., n\}. \tag{26}$$

Hence, in problem (P1), the objective function (23) is maximized under the constraints securing that FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is weak α-$\odot$-consistent, in (24), and $w = (w_1, ..., w_n)$, in (25), is normalized.

If optimization problem (P1) has a feasible solution, i.e. system of constraints (24) - (26) has a solution, then (P1) has also an optimal solution. Let α^* and $w^1 = (w_1^1, ..., w_n^1)$ be an optimal solution of problem (P1). Then α^* is called the *weak $\odot$-consistency grade of FPC matrix* $\tilde{A}$, denoted by $g_\odot(\tilde{A})$, i.e. we define

$$g_\odot(\tilde{A}) = \alpha^*. \tag{27}$$

Here, $0 \leq \alpha^* \leq 1$. Moreover, if $g_\odot(\tilde{A}) = 1$, then FPC matrix $\tilde{A}$ is weak $\odot$-consistent.

If optimization problem (P1) has no feasible solution, which means that FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$ is weak α-$\odot$-consistent for no $\alpha \in [0; 1]$, then we define

$$g_\odot(\tilde{A}) = 0. \tag{28}$$

In that case, the corresponding priority vector will be defined bellow.
Go to Step 2.

Remark 2. In general, problem (P1) is a nonlinear optimization problem that may be solved by a numerical method, e.g. by the well known dichotomy method, which is a sequence of relatively simple optimization problems, see e.g. [3].

In the next step we obtain a corresponding priority vector with our desirable properties.

Step 2.
First, assume that problem (P1) is *feasible*. By solving new optimization problem with α^*-$\odot$-mean matrix $A^* = \{a_{ij}^*\}$ associated to FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$, see (13), we obtain a corresponding priority vector with our desirable properties FS, POP, and RP. Here, the elements of crisp PC matrix $A^* = \{a_{ij}^*\}$ are defined by (13) as

$$a_{ij}^* = (a_{ij}^L(\alpha^*) \odot a_{ij}^R(\alpha^*))^{(\frac{1}{2})},$$

where $\alpha^* \in [0; 1]$ has been calculated in Step 1, $\alpha^* = g_\odot(\tilde{A})$.

Now, we solve problem (P2) as follows. Let $A^* = \{a_{ij}^*\} \in PC_n(\mathcal{G})$ be a PC matrix, $\epsilon > e$. Based on this PCM, we define the following two sets of indexes:

$$I^{(1)}(A^*) = \{(i, j) | i, j \in \{1, \ldots, n\}, a_{ij}^* = e\}, \tag{29}$$

$$I^{(2)}(A^*) = \{(i, j) | i, j \in \{1, \ldots, n\}, a_{ij}^* > e\}, \tag{30}$$

An *error index*, $\mathcal{E}(A^*, w)$, of $A^* = \{a_{ij}^*\}$ and $w = (w_1, \ldots, w_n)$ has been defined by (22) as

$$\mathcal{E}(A^*, w, \alpha^*) = \max_{\{i,j \in \{1,\ldots,n\}\}} \{\|a_{ij}^* \odot w_j \div w_i\|\}. \tag{31}$$

(P2)

$$\mathcal{E}(A^*, w, \alpha^*) \longrightarrow \min; \tag{32}$$

subject to

$$a_{ij}^L(\alpha^*) \leq w_i \div w_j \leq a_{ij}^R(\alpha^*) \text{ for all } i, j \in \{1, ..., n\}, \tag{33}$$

$$\bigodot_{k=1}^{n} w_k = e, w_k \in G \quad \text{for all } k \in \{1, ..., n\}. \tag{34}$$

Problem (P2) is feasible, as $w^1 = (w_1^1, \ldots, w_n^1)$ is a feasible solution of problem (P1), hence, it is also a feasible solution of (P2) with objective function (32). The optimal solution $w^* = (w_1^*, ..., w_n^*)$ of (P2) will be called the $\odot$-*priority vector of* $\tilde{A}$.

Second, assume that problem (P1) is *infeasible*. By solving problem (P2) with $\alpha^* = 0$, where α^*-$\odot$-mean matrix $A^* = \{a_{ij}^*\}$ is associated to FPC matrix $\tilde{A} = \{\tilde{a}_{ij}\}$, see (13), we obtain a corresponding priority vector $w^* = (w_1^*, ..., w_n^*)$ with our desirable properties FS, POP, and RP, i.e. (35), (36). Here, however, the elements of crisp PC matrix $A^* = \{a_{ij}^*\}$ are defined by (13) as

$$a_{ij}^* = (a_{ij}^L(0) \odot a_{ij}^R(0))^{(\frac{1}{2})}.$$

In order to obtain an $\odot$-priority vector satisfying desirable properties FS, POP and RP, we have to solve problem (P2) with two additional constraints

$$w_r = w_s \quad \forall (r, s) \in I^{(1)}(A^*), \tag{35}$$

$$w_r \geq w_s \odot \epsilon \quad \forall (r, s) \in I^{(2)}(A^*). \tag{36}$$

The existence of such optimal solution satisfying properties (35) and (36) is, however, not secured.

The $\odot$-*inconsistency index of* $\tilde{A}$, $I_\odot(\tilde{A})$, is defined as follows.

$$I_\odot(\tilde{A}) = \sup\{\inf\{\mathcal{E}(A^m(\alpha), w, \alpha) | w = (w_1, ..., w_n) \text{ satisfies (25)}\} | \alpha \in [0; 1]\}. \tag{37}$$

Remark 3. If $\alpha^* = 1, w^* = (w_1^*, ..., w_n^*)$ is the optimal solution of (P2), then by (37) and Definition 6 we obtain

$$I_\odot(\tilde{A}) = \mathcal{E}(A^*, w^*, \alpha^*) = e. \tag{38}$$

Moreover, if $\alpha^* < 1$, then $I_\odot(\tilde{A}) > e$.

Remark 4. In general, the uniqueness of optimal solution of (P2) is not saved. Depending on the particular operation $\odot$, problem (P2) may have multiple optimal solutions which is an unfavorable fact from the point of view of the DM. In this case, the DM should reconsider some (fuzzy) evaluations in the original pairwise comparison matrix. Consequently, we obtain the following proposition.

Proposition 5. *If* $\tilde{A} = \{\tilde{a}_{ij}\}$ *is a FPC matrix, then exactly one of the following two cases occurs:*

- *Problem (P1) has a feasible solution* α^*. *Then weak consistency grade* $g_{\odot}(\tilde{A}) = \alpha^*,\ 0 \leq \alpha^* \leq 1$.
  For each α, *such that* $0 \leq \alpha \leq \alpha^* \leq 1$, *FPC matrix* $\tilde{A}$ *is weak* α-$\odot$-*consistent. The associated priority vector* $w^* = (w_1^*, ..., w_n^*)$ *is the optimal solution of (P2) and* $I_{\odot}(\tilde{A}) \geq e$.
- *Problem (P1) has no feasible solution. Then consistency grade* $g_{\odot}(\tilde{A}) = 0$, $\tilde{A}$ *is* $\odot$-*inconsistent, hence* $I_{\odot}(\tilde{A}) > e$. *The associated priority vector* $w^* = (w_1^*, ..., w_n^*)$ *is the optimal solution of (P2) with* $A^* = \{a_{ij}^*\}$ *and*

$$a_{ij}^* = (a_{ij}^L(0) \odot a_{ij}^R(0))^{(\frac{1}{2})}.$$

Remark 5. In particular, if $\tilde{A}$ is weak $\odot$-consistent, then $g_{\odot}(\tilde{A}) = 1$ and by the properties of the distance function, we obtain $I_{\odot}(\tilde{A}) = e$. However, if $\tilde{A}$ is weak $\odot$-inconsistent, then $g_{\odot}(A) = 0$ and $I_{\odot}(\tilde{A}) > e$.
In particular, if $\tilde{A} = \{a_{ij}\}$ is a crisp FPC matrix, then $\odot$-inconsistency index $I_{\odot}(\tilde{A}) = e$, if and only if A is $\odot$-consistent.

Example 7. Consider the multiplicative alo-group $\mathcal{R}_+ = (\mathbf{R}_+, ., \leq)$ with $\odot = .$, see Example 2. Let for three alternatives $\mathcal{C} = \{c_1, c_2, c_3\}$, FPCM $\tilde{A} = \{\tilde{a}_{ij}\}$ be given by triangular fuzzy number elements as follows:

$$\tilde{A} = \begin{bmatrix} (1,1,1) & (1,2,3) & (7,8,9) \\ (\frac{1}{3}, \frac{1}{2}, 1) & (1,1,1) & (3,4,5) \\ (\frac{1}{9}, \frac{1}{8}, \frac{1}{7}) & (\frac{1}{5}, \frac{1}{4}, \frac{1}{3}) & (1,1,1) \end{bmatrix},$$

or, equivalently, by α-cut notation. Each fuzzy set is given by the corresponding family of α-cuts, i.e. intervals. Particularly, for $\alpha \in [0;1]$, we obtain:

$$\tilde{A} = \begin{bmatrix} [1;1] & [1+\alpha; 3-\alpha] & [7+\alpha; 9-\alpha] \\ [\frac{1}{3-\alpha}; \frac{1}{1+\alpha}] & [1;1] & [3+\alpha; 5-\alpha] \\ [\frac{1}{9-\alpha}; \frac{1}{7+\alpha}] & [\frac{1}{5-\alpha}; \frac{1}{3+\alpha}] & [1;1] \end{bmatrix}.$$

Here, $\tilde{A}$ is a 3×3 PC matrix with triangular fuzzy number elements and the corresponding piece-wise linear membership functions.
Priority vector of $\tilde{A}$ is obtained as the optimal solution of problem (P2), particularly, $w^* = (w_1^*, w_2^*, w_3^*) = (2.520, 1.260, 0.315)$.
The corresponding ranking of alternatives is $c_1 > c_2 > c_3$.
Weak consistency grade is $g(\tilde{A}) = \alpha^* = 1.0$, hence, $\tilde{A}$ is weak α-$\odot$-consistent for all $0 \leq \alpha \leq 1$, or, in other words, $\tilde{A}$ is weak $\odot$-consistent.
Inconsistency index $I_{\odot}(\tilde{A}) = \mathcal{E}(A_1^*, w^*, 1) = 1.000$.

6 Conclusion

This paper deals with PC matrices with fuzzy elements.In comparison with PC matrices investigated in the literature, here we investigate PCMs with elements

from abelian linearly ordered group (alo-group) over a real interval. We also define the concept of priority vector which is an extension of the well known concept in crisp case and is used for ranking the alternatives. Such an approach allows for extending various approaches known from the literature. Some numerical examples are presented to illustrate the concepts and derived properties.

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Inducing Useful If-Then Rules from Daily Data about Automatic Ticket Gate Machines by Cumulation and Monotonization

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1 Introduction

Toward the discovery of useful information and knowledge from stored data, techniques for data mining and knowledge discovery have been studied remarkably. In this paper, a rough set-based approach to rule induction is applied to real world daily data about automatic ticket gate machines, in order to mine rules useful for the judgment of necessity of maintenance of the machine. By the rough set-based rule induction method, we obtain simple If-Then rules. However, it is not very easy to induce If-Then rules applicable for various homogeneous data from a given training data because there are too many condition attributes. Then we transform the daily data to cumulative data, i.e., accumulated daily data just after the maintenance day, considering the monotonicity between condition and decision attributes. We evaluate the improvement of induced If-Then rules by the transformation by comparison with If-Then rules induced directly from daily data.

2 The Argument

In this research, we apply a rough set-based rule induction algorithm MLEM2 [1] to the daily data of actual automatic ticket gates. By this application, we explore rules useful for judging the necessity of maintenance of the machines. To this purpose, we induce rules useful for the prediction of the number of errors observed a week later. We demonstrate that the simple and direct application of MLEM2

algorithm does not work well for our purpose. Then we transform daily data to cumulative data from most recent maintenance. Because the cumulation of data, we can assume the monotonicity between cumulative data. From this point of view, we restrict the if-then rules to be induced into those satisfying the monotone relation between premises and conclusions. We demonstrate how the proposed approach works by showing the results of experiments using real world daily data about automatic ticket gate machines. We compare the induced rules in four different approaches: (i) direct application of MLEM2 with intervals of errors, (ii) application of MLEM2 to cumulative data with intervals of errors, (iii) application of MLEM2 to cumulative data with minimum and maximum numbers of errors and (iv) application of MLEM2 with monotonicity restriction to cumulative data with minimum and maximum numbers of errors.

3 Results

We use two datasets observed in Machines A and B, respectively. The error occurs more frequently in Machine B than Machine A. A dataset is used for inducing if-then rules and the other dataset is used for testing the induced rules. We examined both cases. In Table 1 to 4 in the Appendix of these proceedings, the statistics (average and standard deviation) of the number of induced rules, length of the condition, support of induced data as well as the statistics (average and standard deviation) of precision, recall and F-Value of induced data in test dataset are shown for some intervals of errors and for some minimum/maximum numbers of errors.

As shown in Tables 1 to 4 in the Appendix of these proceedings, approaches (ii) and (iii) generally produce better rules than approaches (i) and (iV), respectively. We observe that rules for minimum numbers of errors induced in the dataset of Machine A performs well in the estimation of the minimum numbers of errors in Machine B. On the other hand, rules for maximum numbers of errors induced in the dataset of Machine B performs well in the estimation of the maximum numbers of errors in Machine A. These can be understood from the fact that the error occurs more frequently in Machine B than Machine A.

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Alternative proof on the crossing number of $K_{1,4,n}$

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Abstract

The investigation on the crossing numbers of graphs is very difficult problem provided that an computing of the crossing number of a given graph in general is NP-complete problem. The problem of reducing the number of crossings in the graph is studied not only in the graph theory, but also by computer scientists. The exact values of the crossing numbers are known only for some graphs or some families of graphs. The main aim of the paper is to give the crossing number of join product $G + D_n$ for the connected graph G of order five isomorphic with the complete bipartite graph $K_{1,4}$, where D_n consists on n isolated vertices. The proof of the crossing number of $K_{1,4,n}$ was published by a partially unclear discussion of cases by Ho [4]. In our proof, the idea of cyclic permutations and their combinatorial properties will be used. Due to the mentioned algebraic topological approach, we are able to extend known results concerning crossing numbers for join products of new graphs.

1 Introduction

In this article are used notations and definitions of the crossing numbers of graphs like in [6]. We will often use the Kleitman's result [5] on crossing numbers of the complete bipartite graphs. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{for} \quad m \leq 6.$$

Using Kleitman's result [5], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in [6]. Moreover, the exact values for crossing numbers of $G + D_n$ and of $G + P_n$ for all graphs G of order at most four are given in [11]. It is also important to note that, the crossing numbers

of the join product of graph G with other graphs are known for few graphs G of order five and six in [1], [2], [7, 8, 9, 10], [12, 13, 14, 15, 16, 17, 18] and [20].

The methods presented in the paper are new, and they are based on multiple combinatorial properties of the cyclic permutations. The similar methods were partially used first time in the papers [3] and [13]. In [14], [15], [20] and [21], the properties of cyclic permutations are also verified by the help of software in [19].

2 Cyclic Permutations and Configurations

Let G be the connected graph of order five isomorphic with the complete bipartite graph $K_{1,4}$. We consider the join product of G with the discrete graph on n vertices denoted by D_n. The graph $G + D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \ldots, t_n$, where each vertex t_i, $i = 1, 2, \ldots, n$, is adjacent to every vertex of G. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the five edges incident with the vertex t_i. Thus, $T^1 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{5,n}$ and

$$G + D_n = G \cup K_{5,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{1}$$

In the paper, we will use the same notation and definitions for cyclic permutations and the corresponding configurations for a good drawing D of the graph $G + D_n$ like in [14]. Let D be a drawing of the graph $G + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D as the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i have been defined by Hernández-Vélez, Medina, and Salazar [3]. We use the notation (12345) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, and t_iv_5. We have to emphasize that a rotation is a cyclic permutation. We will separate all subgraphs T^i, $i = 1, \ldots, n$, of the graph $G + D_n$ into three mutually-disjoint subsets depending on how many times the considered T^i crosses the edges of G in D. For $i = 1, \ldots, n$, let $R_D = \{T^i : \mathrm{cr}_D(G, T^i) = 0\}$ and $S_D = \{T^i : \mathrm{cr}_D(G, T^i) = 1\}$. Every other subgraph T^i crosses the edges of G at least twice in D. Moreover, let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D$, where $i \in \{1, \ldots, n\}$. Thus, for a given subdrawing of G in D, any subgraph F^i is exactly represented by $\mathrm{rot}_D(t_i)$.

Since there is only one possible drawing of G, without loss of generality, we can choose the vertex notations of the graph in such a way as shown in Fig. 1. Our aim shall be to list all possible rotations $\mathrm{rot}_D(t_i)$ which can appear in D if the edges of T^i do not cross the edges of G. Since there is only one subdrawing of $F^i \setminus \{v_5\}$ represented by the rotation (1234), there are four possibilities for how to obtain the subdrawing of F^i depending on in which region the edge t_iv_5 is placed. These four possibilities under our consideration are denoted by $\mathcal{A}_k$, for $k = 1, \ldots, 4$. For our purposes, it does not matter which of the regions is unbounded, so we can assume that the drawings are as shown in Fig. 2.

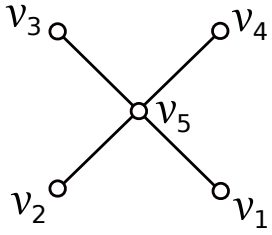

Figure 1: One possible drawing of the graph G.

In the rest of the paper, we represent a cyclic permutation by the permutation with 1 in the first position. Thus, the configurations $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ are represented by the cyclic permutations (12345), (12534), (12354) and (15234), respectively. Of course, in a fixed drawing of the graph $G + D_n$, some configurations from $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$ need not appear. We denote by $\mathcal{M}_D$ the set of all configurations for the drawing D belonging to $\mathcal{M}$.

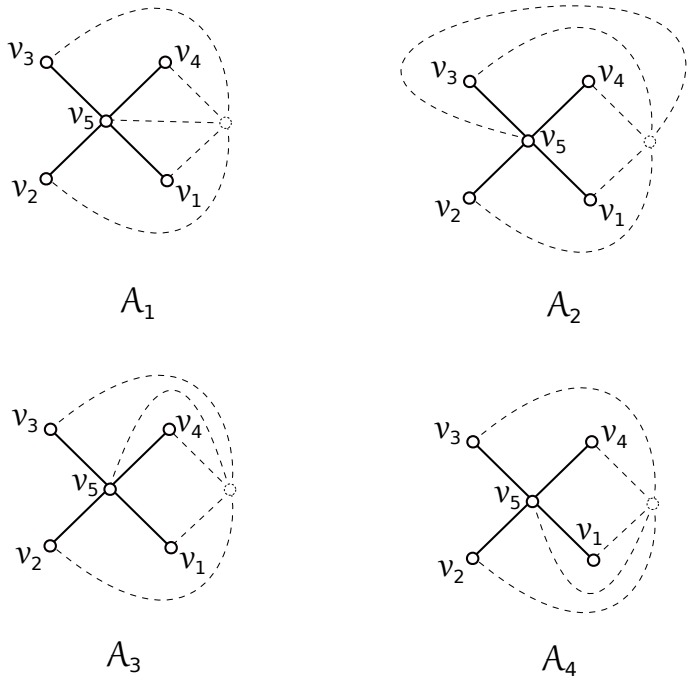

Figure 2: Drawings of four possible configurations from $\mathcal{M}$ of the subgraph F^i.

We remark that if two different subgraphs F^i and F^j with configurations from $\mathcal{M}_D$ cross in a drawing D of $G + D_n$, then only the edges of T^i cross the edges of T^j. Thus, we will deal with the minimum numbers of crossings between two different subgraphs F^i and F^j depending on their configurations. Let $\mathcal{X}$, $\mathcal{Y}$ be the configurations from $\mathcal{M}_D$. We shortly denote by $\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in R_D$ such that F^i, F^j have configurations $\mathcal{X}$, $\mathcal{Y}$, respectively. Finally, let $\mathrm{cr}(\mathcal{X}, \mathcal{Y}) = \min\{\mathrm{cr}_D(\mathcal{X}, \mathcal{Y})\}$ over all good drawings of the graph $G + D_n$ with $\mathcal{X}, \mathcal{Y} \in \mathcal{M}_D$. Our aim is to establish $\mathrm{cr}(\mathcal{X}, \mathcal{Y})$ for all pairs $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$.

Now, we are ready to find the necessary numbers of crossings between subgraphs T^i and T^j for the corresponding configurations of F^i and F^j from $\mathcal{M}$. The configurations $\mathcal{A}_1$ and $\mathcal{A}_2$ are represented by the cyclic permutations (12345) and (12534), respectively. Since the minimum number of interchanges of adjacent elements of (12345) required to produce cyclic permutation $\overline{(12534)} = (14352)$ is two, any subgraph T^j with the configuration $\mathcal{A}_2$ of F^j crosses the edges of T^i at least twice, that is, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_2) \geq 2$. Details have been worked out by Woodall [22]. The same reason gives $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_3) \geq 3$, $\mathrm{cr}(\mathcal{A}_1, \mathcal{A}_4) \geq 3$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_3) \geq 3$, $\mathrm{cr}(\mathcal{A}_2, \mathcal{A}_4) \geq 3$ and $\mathrm{cr}(\mathcal{A}_3, \mathcal{A}_4) \geq 2$. Clearly, also $\mathrm{cr}(\mathcal{A}_i, \mathcal{A}_i) \geq 4$ for any $i = 1, \ldots, 4$. The resulting lower bounds for the number of crossings of configurations from $\mathcal{M}$ are summarized in symmetric Table 1 (here, $\mathcal{A}_k$ and $\mathcal{A}_l$ are configurations of the subgraphs F^i and F^j, where $k, l \in \{1, 2, 3, 4\}$).

$-$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$\mathcal{A}_1$	4	2	3	3
$\mathcal{A}_2$	2	4	3	3
$\mathcal{A}_3$	3	3	4	2
$\mathcal{A}_4$	3	3	2	4

Table 1: The necessary number of crossings between T^i and T^j for the configurations $\mathcal{A}_k$, $\mathcal{A}_l$.

3 The crossing number of $G + D_n$

Two vertices t_i and t_j of $G + D_n$ are *antipodal* in a drawing of $G + D_n$ if the subgraphs T^i and T^j do not cross. A drawing is *antipodal-free* if it has no antipodal vertices. In the rest of the paper, each considered drawing of the graph $G + D_n$ will be assumed antipodal-free. In the proof of the main theorem, the following lemma related to some restricted subdrawings of the graph $G + D_n$ is needful. So, let us first define notation of regions in some subdrawings of $G + D_n$. The unique drawing of F^i contains five regions with the vertex t_i on its boundary. For example, if F^i has the configuration $\mathcal{A}_1$, then let us denote these five regions by $\omega_{1,5,2}$, $\omega_{2,3,5}$, $\omega_{3,5,4}$, $\omega_{4,5}$ and $\omega_{1,5}$ depending on which of vertices are located on the boundary of the corresponding region.

Lemma 1 *Let D be a good and antipodal-free drawing of $G + D_n$, for $n > 2$, with the vertex notation of the graph G in such a way as shown in Fig. 1. Let for $T^i \in R_D$, the corresponding subgraph F^i has the configuration $\mathcal{A}_j \in \mathcal{M}_D$ for some $j \in \{1, \ldots, 4\}$. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 1$, then all possible $rot_D(t_k)$ are given in Table 2.*

$\mathrm{conf}(F^i)$	$\mathrm{rot}_D(t_k)$
$\mathcal{A}_1$	(12543), (15342)
$\mathcal{A}_2$	(13452), (12435)
$\mathcal{A}_3$	(15324), (14523)
$\mathcal{A}_4$	(14235), (13254)

Table 2: The corresponding rotations of t_k for $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 1$.

Proof. Let us assume the configuration $\mathcal{A}_1$ of F^i, i.e., $\mathrm{rot}_D(t_i) = (12345)$. The unique subdrawing $D(F^i)$ of the subgraph F^i contains five regions with the vertex t_i on their boundaries. If there is a subgraph $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 1$, then the vertex t_k must be placed in the region with three vertices of G on its boundary. Therewith, if $t_k \in \omega_{1,5,2}$ or $t_k \in \omega_{3,5,4}$, then the edge $t_k v_4$ or $t_k v_1$ crosses the edges of $G \cup T^i$ at least twice, respectively. In the case $t_k \in \omega_{2,5,3}$, then the edge $v_3 v_5$ or $v_2 v_5$ of G must be crossed by the edge $t_k v_4$ or $t_k v_1$, respectively. This forces $\mathrm{rot}_D(t_k) = (12543)$ and $\mathrm{rot}_D(t_k) = (15342)$. For the remaining possible cases of the configurations of the subgraph F^i, using the same arguments, one can easy to verify the mentioned rotations of the vertex t_k in Table 2. □

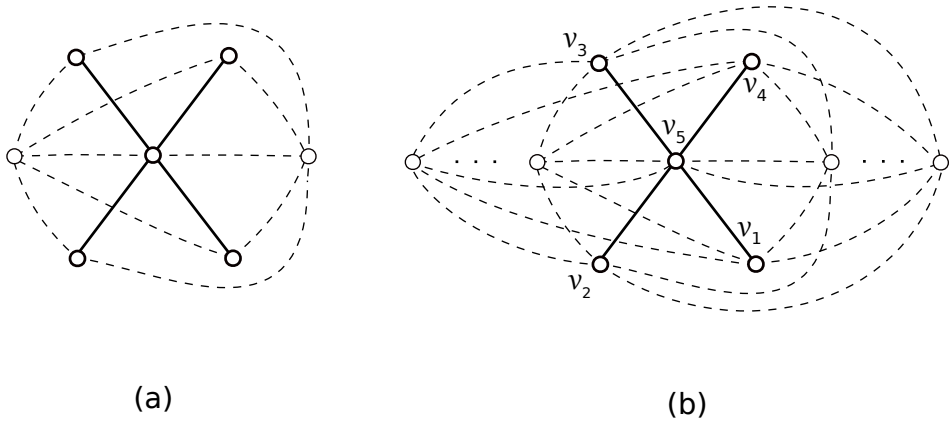

Figure 3: Two drawings of $G + D_2$ and of $G + D_n$.

Theorem 1 $\mathrm{cr}(G+D_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor = n(n-1)$ *for* $n \geq 1$.

Proof. In Fig. 3(b) there is the drawing of $G+D_n$ with $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings. Thus, $\mathrm{cr}(G+D_n) \leq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$. We prove the reverse inequality by induction on n. The graph $G+D_1$ is planar, hence $\mathrm{cr}(G+D_1) = 0$. The graph $G+D_2$ contains a subdivision of $K_{3,4}$, and therefore $\mathrm{cr}(G+D_2) \geq 2$. So, $\mathrm{cr}(G+D_2) = 2$ by the good drawing of $G+D_2$ in Fig. 3(a). Suppose now that, for some $n \geq 3$, there is a drawing D with

$$\mathrm{cr}_D(G+D_n) < 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor \tag{2}$$

and that

$$\mathrm{cr}(G+D_m) \geq 4\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + 2\left\lfloor \frac{m}{2} \right\rfloor \quad \text{for any integer } m < n. \tag{3}$$

We claim that the considered drawing D must be antipodal-free. For a contradiction suppose, without loss of generality, that $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. Using positive values in Table 1 and possible subdrawings in Fig. 2, one can easily to verify that both subgraphs T^n and T^{n-1} are not from the set R_D, and if $T^n \in R_D$ then $\mathrm{cr}_D(G, T^{n-1}) \geq 2$ by possible subdrawings in Fig. 2, i.e., $\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 2$. The known fact that $\mathrm{cr}(K_{5,3}) = 4$ implies that any T^k, $k = 1, 2, \ldots, n-2$, crosses $T^{n-1} \cup T^n$ at least four times. So, for the number of crossings in D we have

$$\mathrm{cr}_D(G+D_n) = \mathrm{cr}_D(G+D_{n-2}) + \mathrm{cr}_D(T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{5,n-2}, T^{n-1} \cup T^n) +$$

$$+\mathrm{cr}_D(G, T^{n-1} \cup T^n) \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2\left\lfloor \frac{n-2}{2} \right\rfloor + 4(n-2) + 2 = n(n-1).$$

This contradiction with the assumption (2) confirms that the considered drawing D must be antipodal-free. Moreover, if $r = |R_D|$ and $s = |S_D|$, the assumption (3) together with the well-known fact $\mathrm{cr}(K_{5,n}) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ imply that in D, there is at least one subgraph T^i, which do not cross the edges of G. More precisely:

$$\mathrm{cr}_D(G) + \mathrm{cr}_D(G, K_{5,n}) \leq \mathrm{cr}_D(G) + 0r + 1s + 2(n-r-s) < 2\left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$s + 2(n-r-s) < 2\left\lfloor \frac{n}{2} \right\rfloor. \tag{4}$$

This forces that $r \geq 1$, and $2r + s \geq 2n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1$. Now, for $T^i \in R_D$, we will discuss the existence of possible configurations of subgraphs $F^i = G \cup T^i$ in the drawing D. Without loss of generality, we can choose the vertex notation of the graph G in such a way as shown in Fig. 1. Thus, we will deal with the configurations belonging to the nonempty set $\mathcal{M}_D$. Let us first denote by $S_D(\mathcal{A}_1, \mathcal{A}_2)$ and $S_D(\mathcal{A}_3, \mathcal{A}_4)$ the set of all subgraphs $T^k \in S_D$ for which there is a $T^i \in R_D$ such that $\mathrm{cr}_D(T^k, T^i) = 1$ with the configuration $\mathcal{A}_1$ or $\mathcal{A}_2$ and $\mathcal{A}_3$ or $\mathcal{A}_4$ of F^i, respectively.

Consequently, let us denote $s_1 = |S_D(\mathcal{A}_1, \mathcal{A}_2)|$, and $s_2 = |S_D(\mathcal{A}_3, \mathcal{A}_4)|$. Remark that $S_D(\mathcal{A}_1, \mathcal{A}_2)$ and $S_D(\mathcal{A}_3, \mathcal{A}_4)$ are disjoint subsets of S_D provided that the corresponding $\mathrm{rot}_D(t_k)$ are given by Lemma 1. Of course, the mentioned subsets can be empty, i.e., $0 \leq s_1 + s_2 \leq s$. In addition, we will discuss two possibilities:

Case 1: $S_D(\mathcal{A}_1, \mathcal{A}_2)$ and $S_D(\mathcal{A}_3, \mathcal{A}_4)$ are empty subsets, i.e., for any $T^i \in R_D$ there is no $T^k \in S_D$ with $\mathrm{cr}_D(T^i, T^k) = 1$.

1. $\{\mathcal{A}_j, \mathcal{A}_{j+1}\} \subseteq \mathcal{M}_D$ for some $j \in \{1, 3\}$. Without lost of generality, let us suppose that $\{\mathcal{A}_1, \mathcal{A}_2\} \subseteq \mathcal{M}_D$. We will discuss two possibilities over congruence n modulo 2.

   - Let n be odd, and let us also consider two different subgraphs T^{n-1}, $T^n \in R_D$ such that F^{n-1} and F^n have configurations $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Then, $\mathrm{cr}_D(T^{n-1} \cup T^n, T^i) \geq 6$ holds for any $T^i \in R_D$ with $i \neq n-1, n$ by summing the values in the corresponding two rows of Table 1. Moreover, it is obvious that the condition $\mathrm{cr}_D(G \cup T^{n-1} \cup T^n, T^k) \geq 5$ is fulfilling for any $T^k \in S_D$. Thus, by fixing the graph $G \cup T^{n-1} \cup T^n$ we have

$$\begin{aligned}\mathrm{cr}_D(G + D_n) &= \mathrm{cr}_D(K_{5,n-2}) + \mathrm{cr}_D(K_{5,n-2}, G \cup T^{n-1} \cup T^n) \\ &+ \mathrm{cr}_D(G \cup T^{n-1} \cup T^n) \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 6(r-2) \\ &+ 5s + 4(n-r-s) + 2 = 4\frac{n-3}{2}\frac{n-3}{2} + 4n + (2r+s) - 10 \\ &\geq (n-3)(n-3) + 4n + 2n - 2\,\frac{n-1}{2} + 1 - 10 \geq n(n-1).\end{aligned}$$

   - Let n be even, and let us also consider that $T^n \in R_D$ with the configuration $\mathcal{A}_1$ of the subgraph F^n, and the number of subgraphs with the associated configuration $\mathcal{A}_1$ is at least as much as the number of subgraphs with the configuration $\mathcal{A}_2$. Hence,

$$\sum_{T^i \in R_D,\ i \neq n} \mathrm{cr}_D(T^n, T^i) \geq 3(r-2) + 2 = 3(r-1) - 1.$$

     So, by fixing the graph $G \cup T^n$ we have

$$\begin{aligned}\mathrm{cr}_D(G + D_n) &\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3(r-1) - 1 + 3(n-r) + 0 \\ &= (n-2)(n-2) + 3n - 4 \geq n(n-1).\end{aligned}$$

2. $\{\mathcal{A}_j, \mathcal{A}_{j+1}\} \not\subseteq \mathcal{M}_D$ for any $j = 1, 3$. Without lost of generality, we can assume that $T^n \in R_D$ with the configuration $\mathcal{A}_k$ of F^n for some $k \in \{1, \ldots, 4\}$. Then $\mathrm{cr}_D(T^n, T^i) \geq 3$ trivially holds for any $T^i \in R_D$ with $i \neq n$. Thus, by fixing the graph $G \cup T^n$ we have

$$\mathrm{cr}_D(G + D_n) \geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3(n-1) + 0 \geq n(n-1).$$

Case 2: At least one of the subsets $S_D(\mathcal{A}_1, \mathcal{A}_2)$ and $S_D(\mathcal{A}_3, \mathcal{A}_4)$ is nonempty. Without lost of generality, let us suppose that $s_1 \geq s_2$. Let us also consider a subgraph $T^k \in S_D(\mathcal{A}_1, \mathcal{A}_2)$, and some $T^i \in R_D$ with $\mathrm{cr}_D(T^k, T^i) = 1$. Then, $\mathrm{cr}_D(T^i \cup T^k, T^j) \geq 5$ holds for any $T^j \in R_D$ with $j \neq i$, and $\mathrm{cr}_D(T^i \cup T^k, T^j) \geq 5$ is true for any $T^j \in S_D \setminus S_D(\mathcal{A}_3, \mathcal{A}_4)$ with $j \neq k$.[1] Hence, by fixing the graph $T^i \cup T^k$ we have

$$\mathrm{cr}_D(G + D_n) = \mathrm{cr}_D(G + D_{n-2}) + \mathrm{cr}_D(T^i \cup T^k) + \mathrm{cr}_D(K_{5,n-2}, T^i \cup T^k) +$$

$$+\mathrm{cr}_D(G, T^i \cup T^k) \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2\left\lfloor \frac{n-2}{2} \right\rfloor + 5(r-1) + 5(s_1 - 1)$$

$$+5(s - s_1 - s_2) + 3s_2 + 3(n - r - s) + 1 + 1 = 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

$$+2\left\lfloor \frac{n-2}{2} \right\rfloor + 3n + (2r + s) + s - s_2 - 8 \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + 2\left\lfloor \frac{n-2}{2} \right\rfloor$$

$$+3n + 2n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1 + 1 - 8 \geq n(n-1),$$

where the inequalities $s - s_2 \geq s_1 \geq 1$ were used.

Thus, it was shown in all mentioned cases that there is no good drawing D of the graph $G + D_n$ with less than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings. This completes the proof of the main theorem. □

4 Four Other Graphs

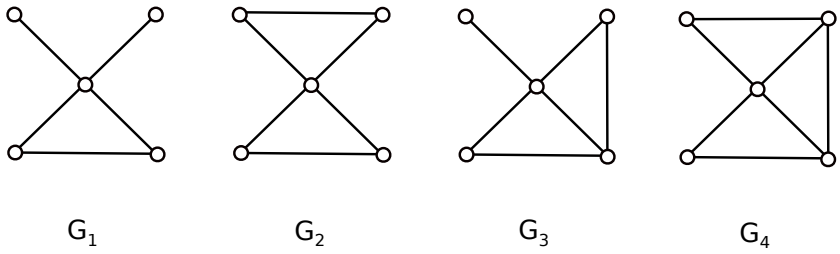

Figure 4: Four graphs G_1, G_2, G_3 and G_4 by adding new edges to the graph G.

Finally, into the subdrawing in Fig. 3(b), we are able to add the edges v_1v_2, v_2v_3 and v_3v_4 to the graph G without additional crossings, and we obtain four new graphs G_i for $i = 1, 2, 3, 4$ in Fig. 4. Therefore, the drawings of the graphs $G_1 + D_n$, $G_2 + D_n$, $G_3 + D_n$ and $G_4 + D_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings are obtained. On the other hand, $G + D_n$ is a subgraph of each $G_i + D_n$, and therefore, $\mathrm{cr}(G_i + D_n) \geq \mathrm{cr}(G + D_n)$ for any $i = 1, 2, 3, 4$. Thus, the next results are obvious.

[1] We have to emphasize that, for the mentioned subgraphs $T^k \in S_D(\mathcal{A}_1, \mathcal{A}_2)$, and $T^i \in R_D$ with $\mathrm{cr}_D(T^k, T^i) = 1$, there is a possibility to find a subdrawing of $G \cup T^i \cup T^k \cup T^j$ in which $\mathrm{cr}_D(T^i \cup T^k, T^j) = 3$ for some $T^j \in S_D(\mathcal{A}_3, \mathcal{A}_4)$.

Collorary 1 $\mathrm{cr}(G_i + D_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 1$, *where* $i = 1, \ldots, 4$.

Moreover, into the drawing in Fig. 3(b) in the cases of graphs G, G_1 and G_2, it is possible also to add $n-1$ edges, which form the path P_n, $n \geq 2$ on the vertices of D_n without another crossing. Thus, the next results are also obvious.

Collorary 2 $\mathrm{cr}(G + P_n) = \mathrm{cr}(G_i + P_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor$ *for* $n \geq 2$, *where* $i = 1, 2$.

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Bayesian Networks for the Analysis of Subjective Well-Being

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Abstract

We use Bayesian Networks to model the influence of diverse socio-economic factors on subjective well-being and their interrelations. The classical statistical analysis aims at finding significant explanatory variables, while Bayesian Networks can also help sociologists to explain and visualize the problem in its complexity. Using Bayesian Networks the sociologists may get a deeper insight into the interplay of all measured factors and their influence on the variable of a special interest. In the paper we present several Bayesian Network models – each being optimal from a different perspective. We show how important it is to pay a special attention to a local structure of conditional probability tables. Finally, we present results of an experimental evaluation of the suggested approaches based on real data from a large international survey. We believe that the suggested approach is well applicable to other sociological problems and that Bayesian Networks represent a new valuable tool for sociological research.

1 Introduction

Bayesian Networks (BNs) [13, 10] are probably the most popular representative from the class of probabilistic graphical models. In this paper we show how BNs can help social scientists to get a deeper insight into a studied problem of their interest. We will use the problem of subjective well-being throughout the paper to illustrate key benefits of the suggested approach.

Although the subjective well-being (SWB) has been researched for decades [4, 6] the debate on its association with the material living conditions still continues and many questions remain unanswered. The people mostly think their happiness and satisfaction are directly linked with the wealth. Modern researchers have also proved that material aspects of life matter, yet their findings are sometimes

surprising. Better understanding of the association between the variables related to the material living conditions such as income, wealth, material deprivation and SWB is valuable.

Typical questions the classical statistical analysis can answer are questions like: what factors have a significant influence on subjective well-being? If regression models are used we may study strengths and signs of the influence of explanatory variables on the dependent variable of our interest. In addition to this analysis, probabilistic graphical models help to get a deeper insight into the interplay of all measured factors and their influence on the variable of the special interest. In BNs the relations are visualized graphically using acyclic directed graphs representing conditional independence relations among variables.

The paper is organized as follows. In Section 2 the concept of subjective well-being is introduced and briefly reviewed. In Sections 3 and 4, the hypotheses of SWB and its association with the variables of material situation are examined using appropriate statistical methods on empirical data. The main original contribution of this paper is presented in Section 5, where BNs are applied to the analysis of SWB. We present two principally different approaches to learning BN structures: one based solely on collected data and on minimization of Bayesian information criteria (BIC) and other where we use an expert version of the PC algorithm to build the model using the expert knowledge of the modeled domain. Special attention is given to learning conditional probability tables (CPTs). The general form of these CPTs, which is commonly used in diverse applications, leads to an undesired and counter-intuitive model inference despite a relatively large dataset used for learning. The main problem is a non-monotone behavior. We show that this problem can be overcome by using appropriate local structure of CPTs – we use Ordinal Logistic Regression (OLR) in Section 6. In Section 7 we evaluate models by measuring how well they fit the data and by measuring their prediction accuracy. We also provide an example of the BN model use in Section 8. We summarize our contribution in Conclusions.

2 Subjective Well-Being

Broadly speaking, SWB is the self-evaluation of one's overall life in positive terms [4]. The concept of SWB has little to do with the objective living conditions, it is determined solely by the subjective assessment.

SWB has two dimensions based on [6]. The affective (or emotional) dimension includes positive and negative moods and emotions (affects). They represent on-line evaluations of events occurring in one's life, whereas the happiness is the surplus of the positive affects over the negative ones. Positive and negative affects are considered to be, in essence, the independent factors. The cognitive dimension of SWB means the judgement of one's satisfaction with the life as a whole as well as with the various life domains, such as job, income, family, leisure etc. Hence, the people high in SWB experience pleasant emotions frequently and unpleasant

Table 1: Model Variables

Abbr.	Description	States
SWB	Subjective Well-Being	{unhappy, fairly unhappy, fairly happy, happy}
PAST	Income compared to own past	{better, the same, worse}
OTHR	Income compared to others	{worse, the same, better}
DEPR	Material deprivation	{none, weakly deprived, deprived}
STRS	Subjective economic strain (ability to make ends meet)	{easily, fairly easily, with some difficulty, with difficulty}
FPRO	Financial problems	{none, minor, major}
HOUS	Housing problems	{no defect, single defect, several defects}
INC	Household income	{low, fairly low, fairly high, high}
CRY	Respondent's country	{C1, C2, C3, C4}

emotions rarely and feel satisfied with the conditions of their lives [5].

Some authors strictly distinguish the happiness from the life satisfaction, where the happiness resulted from the positive experience and the life satisfaction is an outcome of an individual evaluations of discrepancy between material and social aspirations, expectations and achievements. The variable of the subjective well-being in our model incorporates both happiness and life satisfaction components.

The most frequently referred correlates of SWB can be grouped, for example, as follows: demographic factors (age; gender; marital status; religion; physical health), social factors (education; occupation; social relationships), personality factors (extraversion; neuroticism; self-esteem; optimism; purpose-in-life), and wider environmental factors (culture; governance; inflation; unemployment; climate etc.). In this study we consider only factors related to the material situation. In Table 1 we list studied model variables, their brief description, and the number of states of these variables[1].

In the analysis we use data from the third survey of the European Quality of Life Study conducted in 2011 [7]. The survey covers all persons aged 18 and more whose usual place of residence is in the territory of the surveyed countries at the time of the data collection. Only one interview per household. We used the data from four post-communist central European countries – the Czech Republic, Hungary, Poland and Slovakia. These four countries are culturally, geographically, economically, and politically similar. The total of 5,298 respondents from these four countries participated in the survey, out of whom 3,259 complete data vectors are extracted by removing respondents having answered the relevant questions incompletely (613 in the Czech Republic; 586 in Hungary; 1,428 in Poland; and 632 in Slovakia).

[1] Variables SWB and INC were transformed from the original scales using a quantile discretization. The states of variables DEPR, STRS, FPRO, and HOUS are summaries from answers of several questions on the corresponding topic.

Table 2: P-values for the hypothesis of equal means of SWB given values of each explanatory variable.

Pairs of values	1 vs. 2	1 vs. 3	2 vs. 3
INC	0.000	0.000	0.000
OTHR	0.000	0.000	0.000
PAST	0.125	0.000	0.000
DEPR	0.000	0.000	0.000
STRS	0.000	0.000	0.000
FPRO	0.000	0.000	0.629
HOUS	0.000	0.000	0.000
Pairs of values	1 vs. 4	2 vs. 4	3 vs. 4
INC	0.000	0.000	0.000
STRS	0.000	0.000	0.000

3 Basic Statistical Analysis

In [18] a basic statistical analysis of the influence of factors related to the material situation on SWB in four countries of Central Europe was performed. The null hypothesis of equal means of SWB were rejected for all variables[2]. The results are summarized in Table 2 where we present p-values of Welch t-test [19] of equal means of SWB. The results of the tests are presented for each pair of values of every factor variable.

From the table we can see that for almost all explanatory variables the hypothesis of equal means can be rejected except for PAST=1 and PAST=2 and for FPRO=2 and FPRO=3, where means are not significantly different. However, we can conclude that all explanatory variables are significant for SWB since all of them help differentiate between SWB values for at least two of their states. More details can be found in [18].

4 Ordinal Logistic Regression

A natural model for ordinal variables is Ordinal Logistic Regression (OLR) [12]. Since the variable SWB has four states the OLR model of the dependent variable Y representing SWB is defined for $i = 1, 2, 3$ using cumulative distribution functions:

$$P(Y \leq i) \quad = \quad logit^{-1}\left(\zeta_i - \sum_{j \in J} \beta_j \cdot x_j\right) \quad ,$$

[2]In this paper the SWB variable used the original ten points scale.

Table 3: Ordinal logistic regression model.

variables	β_j	std. error	t-value	p-value
CRY2	0.3834	0.1124	3.4118	0.001
CRY3	0.7840	0.0947	8.2815	0.000
CRY4	0.1367	0.1081	1.2644	0.206
INC	0.0870	0.0361	2.4108	0.016
OTHR	0.4660	0.0617	7.5470	0.000
PAST	-0.2709	0.0583	-4.6450	0.000
STRS	-0.2741	0.0451	-6.0728	0.000
DEPR	-0.4960	0.0610	-8.1378	0.000
FPRO	-0.1047	0.0496	-2.1098	0.035
HOUS	-0.2044	0.0466	-4.3879	0.000
intercepts				
ζ_1	-2.3252	0.2977	-7.8105	0.000
ζ_2	-0.8687	0.2949	-2.9460	0.003
ζ_3	0.4056	0.2949	1.3754	0.169

where $\zeta_i, i = 1, 2, 3$ are the intercepts for different values of SWB, $\beta_j, j \in J$ are coefficients of explanatory variables $X_j, j \in J$ taking states x_j. The probability distribution of the dependent variable Y is computed from the cumulative distribution functions as $P(Y = i) = P(Y \leq i) - P(Y \leq i - 1)$ for $i = 1, 2, 3, 4$, where $P(Y \leq 0) = 0$ and $P(Y \leq 4) = 1$.

The coefficients and intercepts of the OLR model learned from SWB data are presented in Table 3. We can see that income relatively higher than income of other people increases probability of higher values of SWB. On the other hand, if the respondent has had a higher income in the past than now then his/her SWB has a higher probability of being lower now. Also, problems specified by variables STRS, DEPR, FPRO, and HOUS imply lower SWB. Since the country variable is not ordinal it is transformed to 3 binary variables taking CRY=1 as the reference value.

5 Bayesian Networks for Subjective Well-Being

A main advantage of BNs is that they represent conditional independence relations graphically. Uncertain relations between variables are modeled using the conditional probability distributions. Hence, BNs enable an efficient encoding of a domain knowledge and improve understanding of complex problems. BNs provide a compact representation of the joint probability distribution.

BNs enable exact probabilistic inference assuming the structure and parameters are estimated correctly – the posterior probability distribution of any variable can

be computed. BNs help answering queries under the uncertainty. As the software for modeling and learning the structure and parameters is available, the complex situations can be modelled with the help of BNs.

The BNs construction includes two main consequent phases: (1) determining the structure and (2) learning the parameters. Determining the structure includes definition of model variables and establishing of directed links among the variables in a network. The structure can be determined based on the expert knowledge or learned from the available data using a structure learning algorithm. In Section 5.1 we use the expert knowledge and in Section 5.2 we learn a BN model from data.

5.1 Expert model

In order to make the process of building the expert model structure systematic we decided to follow the scheme of the PC algorithm [16]. The major difference is that in the standard PC algorithm collected data are used to decide whether a Conditional Independence (CI) statement holds or not while in the expert version of the PC algorithm the expert knowledge is used to decide validity of CI statements for this purpose. If necessary, a detailed review of the SWB literature helped us to reach decisions[3]. The resulting model structure is presented on the left hand side in Figure 1.

5.2 BIC optimal model

Another possibility is to use data to learn a BN model structure. A class of standard model estimation methods is based on finding a model that maximizes the log-likelihood (LL) of data given the model. It is well-known that this often leads to overfitting the training data and results in complex models. Therefore criteria that penalizes complex structures are often used instead. The BIC criterion [15] subtracts from LL a penalty which is proportional to the number of parameters of the BN model $\mathcal{M}$:

$$BIC(\mathcal{M}) \quad = \quad LL(\mathcal{M}) - \frac{1}{2}\kappa \log N \ ,$$

where κ is the number of free parameters in model $\mathcal{M}$ and N is the number of data records.

On the right hand side of Figure 1 we present the structure of a BIC optimal BN model. This model was learned using the Gobnilp tool [1]. Apparently, the BIC greedy search implemented in [9] also results in a model that is equivalent to the BIC optimal one.

[3] A detailed description of the whole process exceeds the scope of this paper. This description is part of a paper currently under review in a journal.

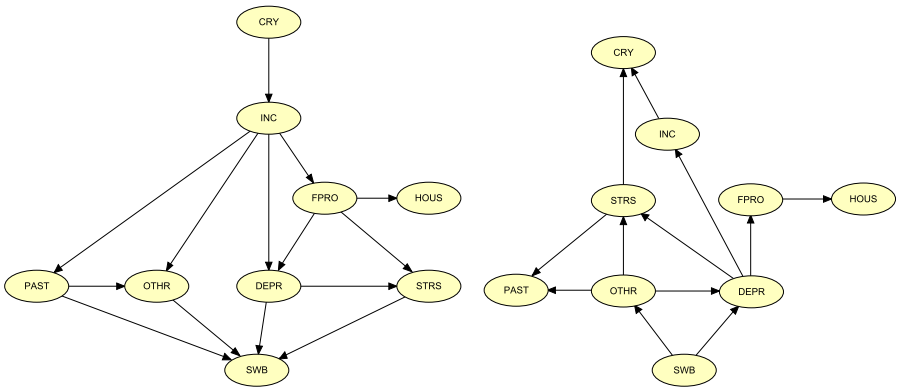

Figure 1: Expert BN (left) and a BIC-optimal BN (right).

6 Conditional probability tables

For the estimation of values of conditional probability tables (CPTs) of BNs from data the EM algorithm is commonly used [3]. If data are complete (i.e., if they contain no missing records) then this procedure reduces to computing relative frequencies from data and this is the case of all our experiments since we used complete data records only. We observed that if the general form of CPTs is used it leads to an undesired and counterintuitive inference despite a relatively large training dataset. The main problem is a non-monotone behavior.

For example, in the model learned from data we observed that $P(\text{SWB} = 1|e, \text{PAST} = 2) \leq P(\text{SWB} = 1|e, \text{PAST} = 3) \leq P(\text{SWB} = 1|e, \text{PAST} = 1)$, which means that the lowest SWB is achieved when variable PAST takes its medium value. One would rather expect the influence of PAST is monotone and SWB is the lowest when the relative income compared to one's own past has become much worse (state 3). The symbol e stands for a particular evidence on remaining parents of SWB. The undesired behaviour was observed for e = (OTHR=1, DEPR=1, STRS=1). Such non-monotone behavior can be observed if there are not enough observations for a given evidence e, which is a quite common situation. This problem should be eliminated since the users do not trust any system with such behaviour. One may believe that this problem disappears when data are large enough but we would like to stress that a mere large dataset does not guarantee that certain combinations are not rare in data. The problem can be properly solved by using a local structure of CPTs appropriate for the application.

Ordinal logistic regression models [12] have several properties that make them good candidates for CPTs in BNs for subjective well-being problem. They assume a natural ordering of the states of variables, which corresponds well to all variables in our model except the country (CRY). Also, the OLR models allow explanatory

Table 4: Comparisons of models' LL and BIC.

Model	LL	BIC std.	BIC OLR
Full	-27,328	-278,930	
OLR	-32,129	-79,419	-32,283
Expert	-29,579	-31,342	
Expert OLR	-29,938	-31,702	-30,112
BIC-optimal	**-29,195**	**-29,822**	
BIC-optimal OLR	-29,395	-30,022	**-29,706**
TAN	-29,285	-30,268	
TAN OLR	-29,525	-30,508	-29,928

variables to have either positive or negative effect on the dependent variable, which fits well the studied problem. We learned OLR models for all CPTs of our expert model except for two CPTs: $P(CRY)$ and $P(INC|CRY)$. In this way, the non-monotonicity property observed for general CPTs completely disappeared.

Other methods that guarantee monotonicity in CPTs exist, e.g. [17, 11, 14]. We have decided to use OLR models since they are commonly used in sociology. However, a more detailed study considering other methods would be interesting, but we leave this task for a future research.

7 Models' Evaluation

We will compare the discussed BN models and the Tree Augmented Naive Bayes model (TAN), which is a BN model commonly used in classification problems [8]. First, we compare the models using the Log-Likelihood (LL) and the Bayesian Information Criteria (BIC). The values of these two criteria are presented in Table 4. The measures are computed with respect to the whole dataset consisting of 3259 data vectors.

From Table 4 we can see that the best model with respect to the BIC criteria is (indeed) the BIC-optimal model whose structure was learned by Gobnilp. The Expert model with unrestricted CPTs has 436 free parameters. When we restrict the conditional probability tables to have parameters of the OLR models (for all nodes except CRY and INC since CRY is not an ordinal variable) the number of free parameters drops to 43. This means the penalty reduces from 1,763 to 174, which implies the BIC value of Expert OLR drops from -31,702 to -30,112. Thus, by the OLR restriction of the CPTs we can get a significantly better BIC value. Similar observations hold for the BIC-optimal and BIC-optimal OLR models.

The primary goal of our work was to help sociologists to get a deeper insight into the problem of SWB, to explain the relation between variables, and to provide a tool for computations of marginal conditional probabilities in situations of sociologists'

interest. However, this being said, we decided to test also the predictive ability of the learned models. The prediction variable is SWB.

We split data into 10 folds and used 10-fold cross-validation to evaluate models predictive abilities[4]. In order to analyze the influence of the data size we performed experiments on fractions of the whole dataset. This means that for small subsets of data we performed more cross-validation experiments. In this way we have achieved comparable results since each single respondent record was used exactly once in testing (in all considered data sizes).

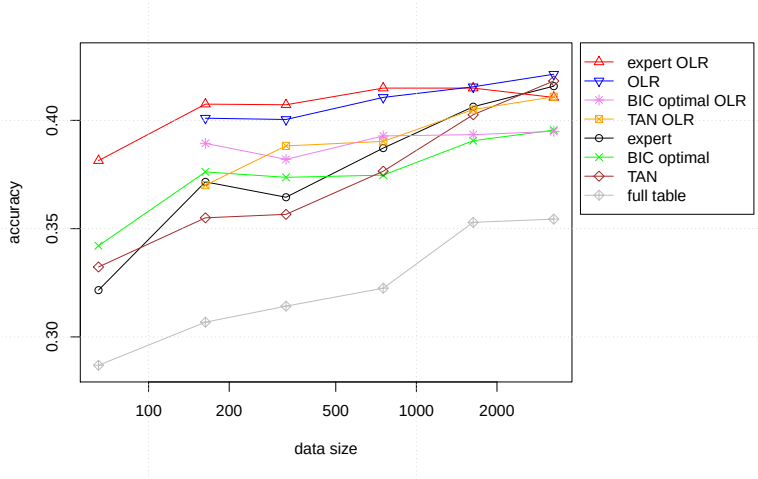

Figure 2: Accuracy with respect to the size of training data.

The models' accuracy is presented in Figure 2. It is the ratio of correctly classified instances with respect to all instances:

$$acc \quad = \quad \frac{1}{N} \sum_{i=1}^{4} C(i,i) \ ,$$

where symbol C denotes the confusion matrix which contains at $C(i,j)$ the number of cases predicted as SWB=i with the reference value SWB=j and N is the total number of instances, i.e.,

$$N \quad = \quad \sum_{i=1}^{4} \sum_{j=1}^{4} C(i,j) \ .$$

From the plot we can see that for small data sizes the OLR versions of all models have better accuracy than their standard versions. The expert OLR and standard

[4]The structure of the tested models was fixed. Each time, nine folds were used to learn model parameters only.

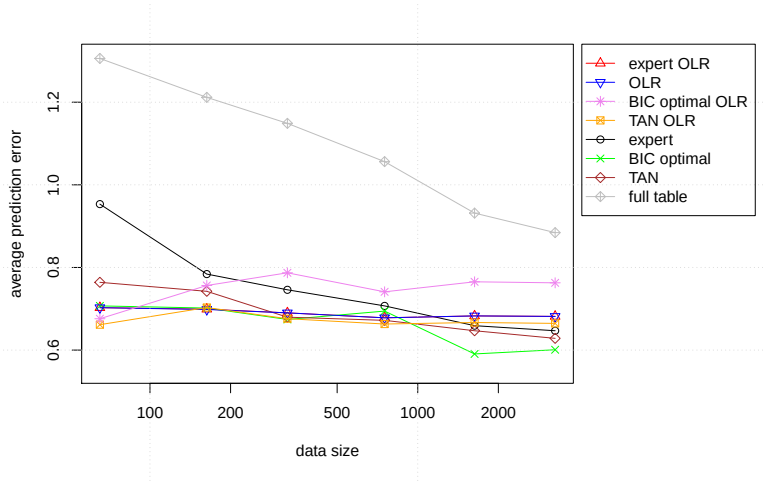

Figure 3: Prediction error with respect to the size of training data.

OLR models performed best. The standard expert model and TAN require more training data to achieve comparable performance. BIC optimal model remains significantly worse than his competitors[5]. The worst of all tested models is the full table model despite this model has the best fit of data. This model is a typical example of a model overfited to training data but performing badly on testing data. Though, the overall accuracy of 0.42 may seem to be low, we should stress that it is significantly higher than the no-information-rate, which is 0.278 (note that SWB has 4 states). The observed level of accuracy is a natural consequence of restricting the study to only factors related to the material situation. Clearly, the omitted factors also play an important role in SWB and since they are not part of our tested models we cannot hope for predictions of a very high accuracy. But, as we have already mentioned, the high accuracy predictions were not the primer goal of our work reported in this paper.

Since the SWB variable is ordinal a more appropriate measure of models' performance than accuracy might be a prediction error. Contrary to the accuracy, which does not consider the distance between the predicted and observed SWB values, the average prediction error defined below does it by means of the absolute difference between the predicted and observed SWB value:

$$err \quad = \quad \frac{1}{N} \sum_{i=1}^{4} \sum_{j=1}^{4} |i-j| \cdot C(i,j) \ .$$

From Figure 3 we can see that for small data sizes the full table model and the (non OLR) expert model are clear losers and TAN is also slightly worse. Other

[5]We verified the claimed differences by Wilcoxon signed-rank tests. Due to the lack of space we do not report the p-values of these tests in this paper.

models have comparable performance. When more training data are available TAN quickly gains comparable performance and latter even the (non OLR) expert model also gains comparable performance to models that were better on smaller datasets. We observe an unexpected performance deterioration for the BIC optimal OLR. We have looked more closely at this model and in its confusion matrix we can see that this model never classifies any instance as SWB=3. Since 3 is one of the middle SWB values the prediction error is affected more than the accuracy. The BIC optimal OLR model is simply not a good candidate for the classification since only two variables (OTHR and DEPR) have any influence on SWB and by restricting the already small CPTs further by the OLR requirement we worsen its performance.

When compared to accuracy the prediction error is more satisfactory, since the prediction error values imply that most testing instances have their difference between the predicted and observed SWB values at most 1. For example, for the best performing model on the largest training set, i.e. for BIC optimal, only 20% of tested instances had this difference larger than 1, while for 40% of instances this difference was equal to 1 and 40% of instances were correctly predicted.

8 An example of a BN model use

In Figure 4 we present an example of a model use. For this purpose we use the Expert OLR model whose structure was presented on the left hand side of Figure 1. This model can be used to predict most probable values of variables of interest in different life circumstances.

For example, assume a person with a low income, but with no subjective material deprivation and making easily ends meet (a low subjective economic strain). When we enter these conditions as evidence into the BN model we can read the conditional probability distributions of remaining variables. From Figure 4 we can see that despite a low income the person is expected to have a high SWB since a low subjective material deprivation and a low subjective economic strain overweight the negative influence of a low income.

9 Related work

Probably, the closest related work is the working paper by Ceriani and Gigliarano [2]. Our motivation is similar to their motivation but our approach differs from their approach in several aspects. Ceriani and Gigliarano do not require monotonicity in CPTs, which we believe is important, as we have shown in our paper. They also used different learning algorithms from the bnlearn R package, that, contrary to Gobnilp, do not necessarily provide BNs optimal with respect to BIC. In addition, we proposed to use an expert-based method which can a domain expert use to build a BN model.

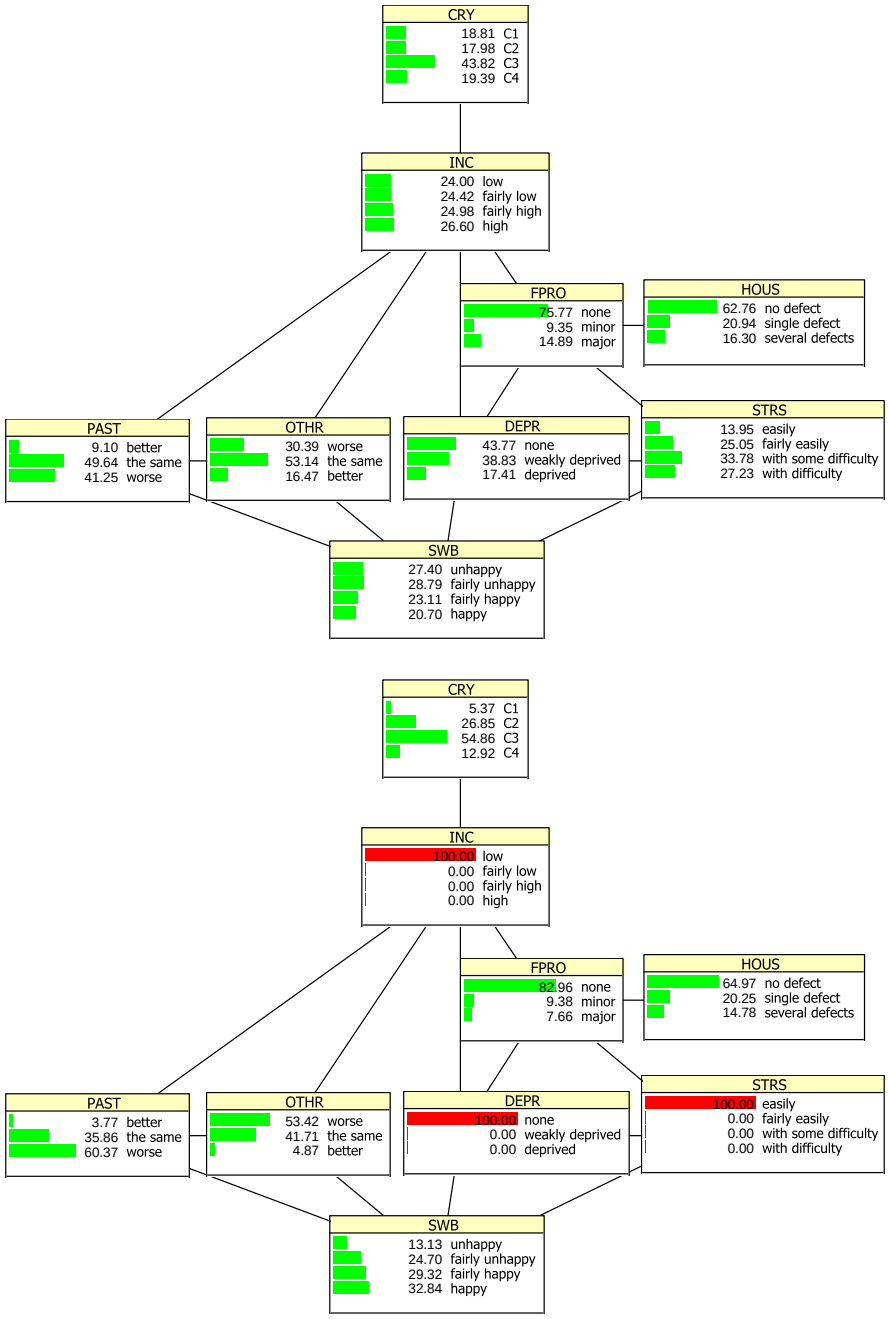

Figure 4: Prior marginal probabilities (top) and the marginals updated for the observed evidence (bottom).

10 Conclusions

In the sociology literature it is still an ongoing debate which factors are important for SWB and which are mediated through others. In the basic statistical analysis of the SWB all studied socio-economic variables were proved to be statistically significant for SWB, but this analysis cannot decide whether their influence is direct or mediated through other variables. We applied BNs to this problem since they can model complex relations between variables. The expert model constructed using an expert version of the PC algorithm can be used to resolve this debate in a systematic and mathematically rigorous way.

From the point of view of sociology, both, the Expert and the BIC-optimal BNs suggest that the objective conditions such as the income and the financial problems influence SWB only indirectly through the subjective perception of the relative income, the material deprivation and the economic stress. We were able to derive this conclusion (and few others) due to the analysis based on BNs. We believe that BNs represent a valuable tool for social scientists.

Acknowledgement

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Optimal Design of Bi-objective Reliable Network Using Genetic Algorithm

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Abstract

Internet, power network, traffic network, etc. are network systems that are required high reliability. Independently, costs may occur for the construction or maintenance of links in the networks. In this study, we consider a bi-objective network design problem with objectives of maximizing all-terminal reliabilities and minimizing construction costs of edges. In general, these objectives are in trade-off relation, and cannot be optimized simultaneously. Therefore, the meaning of solving the problem is to find the set of all Pareto solutions. On the other hand, the problem of evaluating all-terminal reliability of a given network is computationally intractable, which suggests that our bi-objective network design problem is computationally intractable as well. Hence, it is reasonable to switch our goal to develop a solution method for finding a set of quasi-Pareto solutions. Previous study restricted calculated networks by using Pareto solutions properties. These Pareto properties were also applied to GA-based algorithm. This previous GA expanded calculatable networks. But solution search of this algorithm is inaccurate. In this paper, we improve previous GA-based algorithm such that obtained non-dominated solutions are closer to Pareto solutions. The accuracy of our proposed algorithm is evaluated based on comparison of quasi-Pareto solutions to other

algorithms.

1 Introduction

Internet, power network, traffic network, etc. are network systems that are required high reliability. In addition, independently, costs may occur for the construction or maintenance of links in the networks. In this study, we consider a bi-objective network design problem with objectives of maximizing all-terminal reliabilities and minimizing construction/maintenance costs of edges. In general, costs occur to increase reliabilities. So, these objectives are in trade-off relation, and cannot be optimized simultaneously. Therefore, the meaning of solving the problem is to find the set of all Pareto solutions. On the other hand, the problem of evaluating all-terminal reliability of a given network is computationally intractable[2], which suggests that our bi-objective network design problem is computationally intractable as well. Akiba et al.[1] proposed an algorithm which finds all Pareto solutions. However, this algorithm can be applied to only networks of restricted size. Hence, it is reasonable to switch our goal to develop a solution method for finding a set of quasi-Pareto solutions, i.e., a set of pairwise non-dominated good solutions in terms of our objectives.
Previous study[3] restricted calculated networks by using Pareto solutions properties. This algorithm can search $n \leq 6$. These Pareto properties were also applied to GA-based algorithm [4]. This previous GA expanded calculable networks. But solution search of this algorithm is inaccurate in $n \geq 6$.
In this paper, we improve previous GA-based algorithm[4] such that obtained non-dominated solutions are closer to Pareto solutions in $n \geq 6$. Proposed GA reconsiders selection and crossover operation such that Pareto solutions properties are reflected. And then, the accuracy of our proposed algorithm is evaluated based on comparison of quasi-Pareto solutions to other algorithms.

2 Model Description and Properties of Pareto Solutions

2.1 Notations

For describing our problem, we define some notations as follows.
n : the number of nodes
m : the number of edges of complete graph with n nodes, that is $m = {}_nC_r$.
e_i : the i-th undirected edge
E : the set of undirected edges
x_i : binary variable where, $x_i = 1$ if e_i is included in network, and $x_i = 0$ if not.
$\mathbf{x}$: the vector of binary variable x_i. particularly, $\mathbf{x}$ indicates a connected network.
X : the set of networks
p_i : reliability of edge e_i

c_i : construction cost of edge e_i
$R(\mathbf{x})$: all-terminal reliability of network $\mathbf{x}$
$C(\mathbf{x})$: total construction cost of network $\mathbf{x}$
$\mathbf{x}_k$: the connected network with n nodes by k edges, that is, $\sum_{x_i \in \mathbf{x}} x_i = k$.
X_k : the set of network $\mathbf{x}_k$
$\mathbf{px}_k$: the network which satisfies definition of Pareto solutions in X_k.
PX_k : The set of network $\mathbf{px}_k$

2.2 Assumptions

The following assumptions are considered for the study.

- All-terminal nodes do not fail.
- Each edge is in either operational or failure state.
- Each edge fails independently.
- Each edge e_i has cost ci and reliability p_i, and these values are known.

2.3 Definition of Problem

In this study, we consider network design problem with all-terminal reliability and construction cost. The problem can be expressed as follows.

$$\begin{aligned} R(\mathbf{x}) &\rightarrow max \\ C(\mathbf{x}) &\rightarrow min \\ s.t. \mathbf{x} &\in X \end{aligned}$$

It is a rare case that a network makes two objectives optimal simultaneously. For solving this problem, we find Pareto solutions. Let X be the set of feasible solutions.

- For $\mathbf{x}, \mathbf{x}' \in X$, we say that $\mathbf{x}$ is dominated by $\mathbf{x}'$ if $\mathbf{x}$ satisfies both of following conditions:
  - both $R(\mathbf{x}) \leq R(\mathbf{x}')$ and $C(\mathbf{x}) \geq C(\mathbf{x}')$
  - at least one of $R(\mathbf{x}) < R(\mathbf{x}')$ and $C(\mathbf{x}) > C(\mathbf{x}')$
- We say that $\mathbf{x}$ is a Pareto solution if $\mathbf{x}$ is not dominated by any other solutions.

2.4 Properties of Pareto Solutions

2.4.1 The distribution of Pareto Solutions Candidates

When Pareto solutions are plotted on two-dimensional space, whose axes are $R(\mathbf{x})$ and $C(\mathbf{x})$, Pareto solutions distribution is like an exponential curve. Takahashi et

al.[3] restricted Pareto solutions candidates to near the Pareto front based on the slope of certain linear function. The certain linear function satisfies the following property.

Property 1 :
For $\forall \mathbf{px}_k \in PX_k$, define $\mathbf{px}_k^* = \arg\min \frac{R(\mathbf{px}_k)}{C(\mathbf{px}_k)}$. $\mathbf{px}_{k+1} = (\in PX_{k+1})$ is likely to be constructed by adding an edge to $\mathbf{x}_k$ such that $\frac{R(\mathbf{px}_k^*)}{C(\mathbf{px}_k^*)} < \frac{R(\mathbf{x}_k)}{C(\mathbf{x}_k)}$.

2.4.2 Edge Efficiency

Takahashi et al.[3] thought of an edge which is added to selected networks $\mathbf{x}_k$. Pareto solutions depend on reliability p_i and cost c_i of each edge which constructs the network. The following index is defined.

Definition :
$f_i = (p_i/c_i)$ denotes efficiency of edge e_i.

3 Proposed Algorithm

3.1 Genetic Algorithm

Genetic Algorithm (GA) is based on the mechanism of the natural selection. Solution candidates are represented by genes and are named individuals. The individual with good fitness value is preferentially selected. The crossover and mutation are operated on selected individuals. The repetition of these operations generates solutions.

3.1.1 Initial Population

Initial population is generated by random uniform number, and complete graph $\mathbf{x}^*$ is also generated as a solution. Complete graph is used for selection operation.

3.1.2 Evaluation

Solutions fitness is evaluated by all-terminal reliability and construction cost. All-terminal reliability $R(\mathbf{x})$ is calculated by improved factmem[2], and construction cost $C(\mathbf{x})$ is summation of edges in $\mathbf{x}$.

3.1.3 Selection

This operation selects individuals as parents. Based on Takahashi et al.[3], the property with the distribution of Pareto solutions is used in proposed algorithm. The algorithm based on Pareto properties[3] divides networks into groups according to the number of edges, and restricts Pareto candidates in networks with same

number of edges the order of the number of edges. However, our algorithm generates networks regardless of the number of edges. Therefore, we apply this property in each generation. Previous GA[4] generates selection area by using two solutions. This algorithm generates complete graph $\mathbf{x}^*$ as Initial population. In each generation, Pareto solutions (non-dominated solutions) are obtained. We calculate the line which connects two endpoints $(\mathbf{x}_{min}, \mathbf{x}^*)$ of Pareto solutions . The area for parents selection is above this line, which is shown in Figure 1(a). If minimal cost of non-dominated solution decreases, $\mathbf{x}_{min}$ is updated.
On the other hand, proposed GA generates selection area by using three solutions. This algorithm also generates complete graph $\mathbf{x}^*$ as Initial population. In each generation, Pareto solutions (non-dominated solutions) are obtained. In addition to two endpoints, $\mathbf{x}_{min}$ and $\mathbf{x}^*$, we define a middle restricting point $\mathbf{x}_{mid}$ of Pareto solutions. $\mathbf{x}_{mid}$ is $|PX|/2$-th Pareto solution in ascending order of cost, where PX is the set of Pareto solutions. We calculate two lines which connect $\mathbf{x}_{min}, \mathbf{x}_{mid}$ and $\mathbf{x}_{mid}, \mathbf{x}^*$. The selected parents are above the calculated lines as shown in Figure 1(b). And then, these lines are updated when the minimal cost of non-dominated solutions decreases, or the restricting point $\mathbf{x}_{mid}$ is dominated by other solution.

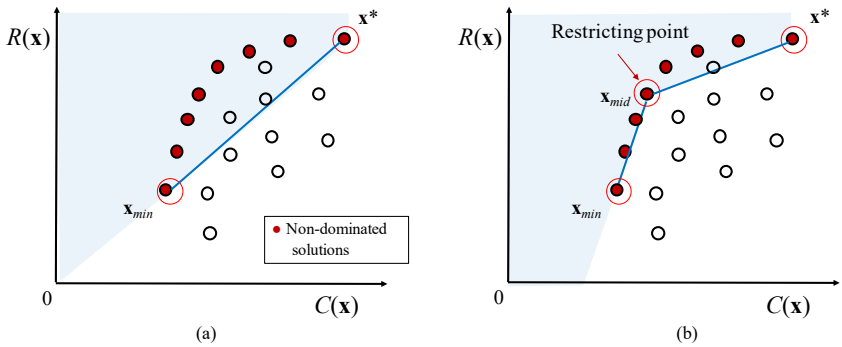

Figure 1: Selection Area of Previous and Proposed GA

3.1.4 Crossover

Individuals are represented by M-dimensional vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, where x_i corresponds to the i-th gene. x_i is a binary variable. $x_i = 1$ if solution has edge e_i, $x_i = 0$ otherwise. The crossover operates on selected two parents' genes and generates two offsprings by swapping their genes. We apply edge efficiency[3] to the crossover operator. Our crossover method is such that each offspring includes edge e_i with higher f_i.

Previous GA[4] generates offsprings in two ways. If one of parents has the most efficient edge, the most efficient edge is crossover point and one-point crossover is conducted such that both include the most efficient edge. Else, usual one-point

crossover is conducted. Crossover point is selected at random (See Figure 2).

Proposed GA expands a way of crossover. If one of parents has more efficient edge, this efficient edge is crossover point and one-point crossover is conducted such that both include the efficient edge. Else, if both parents have (dont have) more efficient edge, usual two-point crossover is conducted. In this case,crossover points are selected at random.

In figure 3(a), since either of parents has edge e_3 with the highest f_i, crossover point is x_3. Right genes of this crossover point are swapped between the parents. On the other hand, in figure 3(b), both parents dont have edge e_3, but one of them has the second most efficient edge e_8. So, x_8 is crossover point. Figure 3(c) shows the case that both parents have (dont have) more efficient edge.

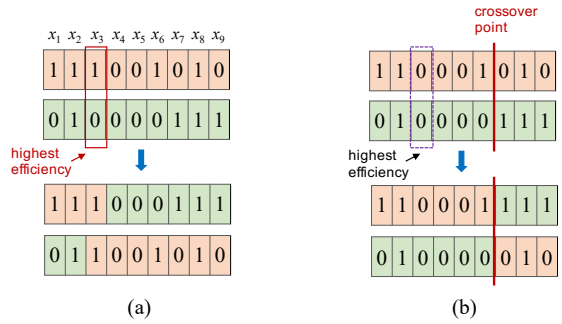

Figure 2: Crossover of Previous GA

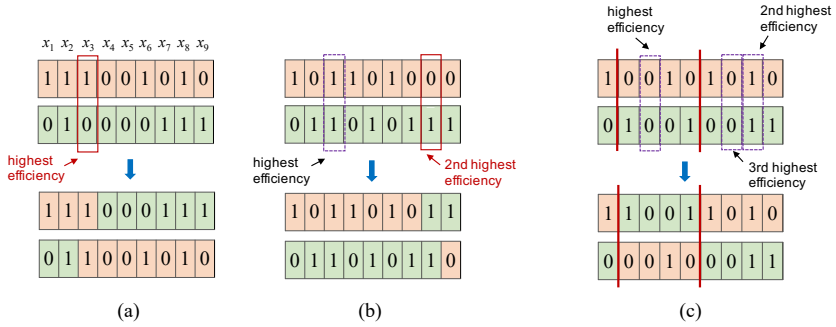

Figure 3: Crossover of Proposed GA

3.1.5 Mutation

The mutation operation changes genes of an offspring with a given rate. Our algorithm decides whether each gene mutates for all offsprings. Then, mutation

may occur several times on one individual.

3.2 Proposed Algorithm

This section describes the procedure of proposed algorithm. In the following procedure, N is population size and G is generation number. X is the current population whose elements are N individuals. X_p is the set of non-dominated solutions in X and PX is the set of non-dominated solutions up to current generation. $\mathbf{x}^*$ is the complete graph.

STEP 1 : (Initialization)

STEP 1-1 : Set $G_{num} \leftarrow 0$, $N_{num} \leftarrow 0$ and $X \leftarrow \emptyset$.

STEP 1-2 : Generate an individual $\mathbf{x}^*$ where $x_i = 1$ for all i.
Generate $N-1$ individuals $\mathbf{x} \in X$ whose genes are obtained by the uniform random number.

STEP 2 : (Fitness evaluation)

STEP 2-1 : Calculate $C(\mathbf{x})$, $R(\mathbf{x})$ for $\forall \mathbf{x} \in X$.

STEP 2-2 : Search for Pareto solutions $X_p \in X$, and $PX \leftarrow PX \cup X_p$

STEP 3 : (Generation of new population)

STEP 3-1 : (Selection)
Select individual $\mathbf{x}_{min}$ and $\mathbf{x}_{mid}$ in the non-dominated solutions PX. $\mathbf{x}_{min}$ satisfies $C(\mathbf{x}_{min}) = min\{C(\mathbf{x}) \mid \forall \mathbf{x} \in PX\}$. For $\forall \mathbf{x}' \in X$, $X \leftarrow X \setminus \mathbf{x}'$ where $\mathbf{x}'$ satisfies neither of following conditions.

- $(R(\mathbf{x}')-R(\mathbf{x}_{min}))/(C(\mathbf{x}')-C(\mathbf{x}_{min})) > (R(\mathbf{x}_{mid})-R(\mathbf{x}_{min}))/(C(\mathbf{x}_{mid})-C(\mathbf{x}_{min}))$.
- $(R(\mathbf{x}^*) - R(\mathbf{x}'))/(C(\mathbf{x}^*) - C(\mathbf{x}')) > (R(\mathbf{x}^*) - R(\mathbf{x}_{mid}))/(C(\mathbf{x}^*) - C(\mathbf{x}_{mid}))$.

STEP 3-2 : (Crossover)
Select $\mathbf{x}_{p1}, \mathbf{x}_{p2} \in X$.
If the value of x_i relating to more efficient edge e_i is equal in $\mathbf{x}_{p1}, \mathbf{x}_{p2}$, two-point crossover is operated.
Otherwise, select a crossover point x_i relating to more efficient edge e_i. Then, two offsprings are generated by swapping genes of parents such that efficient gene $x_i = 1$ in $\mathbf{x}_{p1}, \mathbf{x}_{p2}$.
If no crossover is operated, parents are copied as offsprings.

STEP 3-3 : (Mutation)
With a mutation rate, change genes of offsprings.

STEP 4 : Add generated offsprings to population X' and $N_{num} \leftarrow N_{num} + 2$. If $N_{num} < N$, go to STEP 3-2. Otherwise, go to STEP 5.

Table 1: GA parameters

	N	G	P_c	P_m
Proposed GA	150	3000	0.8	0.01
Previous GA[4]	100	100	0.9	0.03

STEP 5 : $X \leftarrow X'$, $X' \leftarrow \emptyset$. If $G_{num} < G$, $G_{num} \leftarrow G_{num} + 1$ and go to STEP 2. Else, go to STEP 6.

STEP 6 : Output all individuals $\mathbf{x} \in PX$ as Pareto solutions.

4 Numerical Experiments

Numerical experiments are conducted to evaluate proposed algorithm with respect to the robustness. Networks of $6 \leq n \leq 9$ are used as examples. In $n = 6$, proposed algorithm is compared with Akiba et al.[1] whose algorithm enumerates all solutions. In more than $n = 6$, our algorithm is compared with Takahashi et al.[3][4]. Reliability p_i of each edge is generated by random uniform number between 0.70 and 0.99. Cost c_i of each edge is also generated by random uniform number between 70 and 90.

To apply the GA, several GA parameters, which are generation number (G), population size (N), crossover rate (P_c) and mutation rate (P_m), should be determined. We conducted preliminary experiments. As the results, determined parameters are shown in Table 1. Table 1 also shows parameters of Previous GA[4]. In proposed algorithm, optimal solutions are improved by increasing generation number.

In $n = 6$, algorithms are evaluated by obtained rate and error rate. Obtained rate means the percentage of obtained optimal solutions by our algorithm to real Pareto solutions. Proposed algorithm also outputs some non-Pareto solutions as optimal solutions. Error rate means the percentage of these non-Pareto solutions by proposed algorithm to real Pareto solutions. Results of experiments are shown in Table 2. Proposed GA is conducted three times. Obtained rate of proposed GA is more than 90%, and error rate is less than 2%. Proposed GA can find Pareto solutions more correctly than previous studies. In terms of computing time, previous studies are more efficient than proposed algorithm. In cases of more than $n = 6$, proposed GA algorithm is compared with Takahashi et al.[3][4]. In $n = 7$, proposed GA finds better solutions than previous GA. Obtained solutions of proposed GA are close to solutions of Pareto properties[3]. With respect to computing time, Pareto Properties algorithm takes 77.39 seconds, while proposed algorithm takes 1313.25 seconds. Figure 4 shows optimal solutions distribution in $n = 9$. In this case, generation number is set to 1000. Proposed GA finds superior solutions than previous GA. But, proposed GAtakes about five times as long as previous GA.

Table 2: Comparison of Evaluation Criteria

	Computing time(sec.)	Obtained rate(%)	Error rate(%)
Pareto properties[3]	0.55	70.00	17.65
Previous GA[4]	1.67	35.00	52.27
Proposed GA1	154.90	95.00	1.72
Proposed GA2	145.75	96.67	0
Proposed GA3	166.89	95.0	0

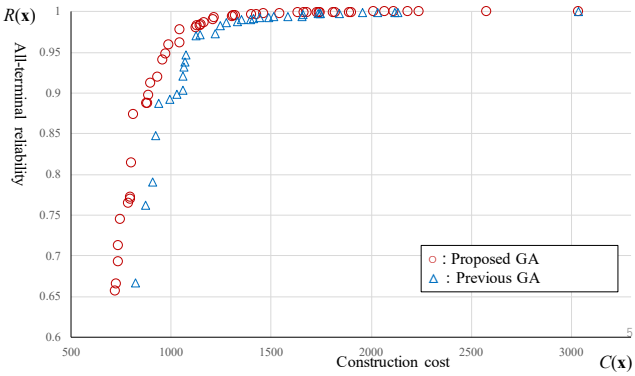

Figure 4: Optimal Solutions of Previous and Proposed GA

5 Conclusion

In our study, we focused on bi-objective network design problem with all-terminal reliability and construction cost. We improved a GA-based algorithm. As the results, proposed algorithm could obtain better solutions than previous GA. Preliminary experiments suggested that optimal solutions of proposed algorithm were improved by increasing generation number and population size. Therefore, computed solutions were increased. Considering computing time, proposed algorithm is inefficient. We may need to restrict gene generation to effective gene. In this study, area for parents selection is more restricted than previous GA. This restriction may reduce obtained optimal solutions per one generation.

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Inducing k-anonymus Binary Cassification Rules from Data Tables

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1 Introduction

When publishing a data table, the privacy protection is indispensable. In this study, we consider the publishment of rules induced from a data table. Rules are generalizations of data so that they are easier to protect the privacy, and by using them we may obtain reproduce the data table to a certain extent if rules are accurate and exhaustive. However, if the number of objects matched to a rule is very small, some private data can be revealed. Therefore, we should take care of privacy protection even on rules induced from data tables. We consider rules inferring a class to which the object belongs. For privacy protection of rules, we apply the k-anonymization technique [1]. Applying k-anonymization technique usualy deteriorates the usufulness of rules or decreases the number of publishable rules. To avoid this, we consider imprecise rules [2]. Inuiguchi et al. [2] have succeeded to induce k-anonymous imprecise rules without great deterioration of usefulness and the number of publishable rules. However, imprecise rules can be induced from data table with more than two classes. Then we investigate a method for inducing k-anonymous imprecise rules from a data table with two classes.

2 Class refinement

We describe a method to induce k-anonymous rules from a decision table with two decision classes D_1 and D_2. First, for each D_i, we induce rules R_{ij} of the form "if u satisfies P_{ij} then u belongs to D_i", $j = 1, 2, \ldots, q_i$ by MLEM2, where q_i is the number of rules induced by MLEM2 for D_i. We define a subset Y_{ij} of object set U whose elements satisfy the condition P_{ij} of rule R_{ij} and its complement N_{ij}, i.e.,

$$Y_{ij} = \{u \in D_i \mid u \text{ satisfies } P_{ij}\}, \quad N_{ij} = U - Y_{ij}. \tag{1}$$

Table 1: Experimental result

	MLEM2	$k=2$	$k=5$	$k=8$	$k=10$
A	0.6842 $\pm$0.1350	0.7325$\pm$0.1186**	0.7425 $\pm$0.1106**	0.7350 $\pm$0.1216**	0.7375 $\pm$0.1192**
D	0.7870 $\pm$0.1347	0.7760 $\pm$0.1335	0.7870 $\pm$0.1301	0.8080 $\pm$0.1155*	0.8070 $\pm$0.1185*
G	0.6550$\pm$0.1472	0.6400$\pm$0.1523	0.6450 $\pm$0.1645	0.6780 $\pm$0.1610	0.6910 $\pm$0.1550**

Using Y_{ij}, a sub-class D_{ij} of D_i is defined by

$$D_{ij} = Y_{ij} - \bigcup_{k=1,2,\ldots,j-1} Y_{ik}, \quad \left(\bigcup_{k=1,0} Y_{ik} = \emptyset \right). \tag{2}$$

Regarding D_{ij} as a decision class, we apply the k-anonymous rule inducing method [2], where we input N_ij as the set of negative examples.

The k-anonymous rule induction method proposed by Inuiguchi et al. [2] can require $o(2^n)$ iterations by its nature. The reducing n is effective to reduce the computational effort. Then, we propose the following simplified algorithm for k-anonymous rule induction.

1. $R := \emptyset$, $i := 1$, $U' := U$, $p := 1$. (In R, we obtain k-anonymous rules.)
2. Induce rules for unions of p classes D_{ij} from objects in $D_{ij} \cap U'$ by MLEM2.
3. The induced rules whose support is not less than k are added to R.
4. Exclude objects satisfying the condition of a rule in R from U'. Let $i := i + 1$ and $p := i$. If $i = n$ or $U' = \emptyset$, terminate the algorithm. Otherwise, return to 2.

3 Numerical experiments

Using datasets 'adult(A)', 'default(D)' and 'german_credit(G)' obtained from UCI Data Repository [3], we examined the performance of the proposed algorithm. In dataset A, using only numerical attributes, we generated 10 decision tables with 120 objects selected randomly. In datasets D and G, all attributes were used to generate 10 decision tables with 100 objects selected randomly. We set $k = 2, 5, 8, 10$. The classification accuracy was evaluated by the 10-fold cross validation method. The obtained results ([average]$\pm$ [standard deviation]) are shown in Table 1.

In Table 1, the column 'MLEM2' shows the classification accuracy of rules induced by MLEM2 without k-anonimization. The significant difference at level of 5% compared to MLEM2 by t-test is shown by $*$ and that at level of 1% is shwon by $**$. The accuracies of the anonymous rules for all k are sufficiently high. In our experiments, the classification accuracy increases as k increases.

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An Overview of Approaches Evaluating Intelligence of Artificial Systems

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Abstract

The goal of Artificial General Intelligence is to create systems capable of solving many different and unforseen tasks. Suitable methods to evaluate the intelligence of artificial systems are thus needed. This review paper searches for such methods. An extensive literature overview is conducted that covers philosophical and cognitive presumptions of intelligence and also definitions and tests of intelligence grounded in Algorithmic Information Theory. A comparison of the introduced approaches identifies two distinct groups based on fundamentally different presumptions. One group of approaches, such as the Turing test, is based on the presumption that success in a complex task is sufficient for intelligence evaluation, while the other group of approaches, such as the Algorithmic Intelligence Quotient test, also require explicit verification of success in simple tasks. This paper concludes that the Algorithmic Intelligence Quotient test is currently the most suitable candidate for a method to evaluate the intelligence of artificial systems.

1 Introduction

The goal of artificial intelligence (AI) as a field of study can be understood differently, cf. [26, 8]: What has been traditionally called a *Weak AI*, or more recently a *Narrow AI*, seeks to create useful tools to solve particular tasks. Such systems may implement a specific cognitive ability, however, the ability is implemented independently from others that are usually considered to manifest in intelligent entities. By contrast, *Artificial General Intelligence (AGI)*, or more traditionally *Strong AI*, seeks to create an artificial system capable of solving many different and possibly unforeseen tasks thus its intelligence is comparable to that of a human.

This review paper looks specifically at *AGI*. One of the crucial problems of *AGI* is the search for suitable methods that can practically evaluate the intelligence of (not only) artificial systems [12]. The goal of this paper is to provide an overview and comparison of such methods.

In an attempt to clarify the problematic concept of intelligence, Section 2 focuses on the evaluation approaches originating from the field of *philosophy and cognitive science*. Such an endeavor as AGI, however, requires more solid methods that can evaluate whether a system is intelligent, and to what extent. Section 3 searches for such methods among approaches grounded in *Algorithmic Information Theory*. Section 4 will conclude the paper with a comparison of the introduced approaches.

2 Philosophy and Cognitive Science

To illustrate the extent of the concept of intelligence, it will be approached along two lines. In Section 2.1, several attempts to answer the question: "What does it mean to think?" will be summarized. In Section 2.2, the relationship between intelligence and other cognitive abilities will be examined.

2.1 Turing Test and its Extensions

The question "What does it mean to think?" is central to the field of AI, as its main goal is the construction of thinking machines . Attempts to answer this question can be traced back to long before the conception of AI. According to Descartes [5], thinking is connected with language and rational speech. Descartes also considers thinking to be universal, i.e. capable of solving numerous different problems.

Turing [32] famously transformed the initial question: "Can machines think?" into the imitation game that is now known as *the Turing Test*. A human interrogator asks questions to two test subjects in order to determine which is a human and which a machine. Indirect communication employed by this setting ensures the test focuses on the differences in thought manifestation instead of superficial characteristics of the test subjects. If the interrogator is unable to determine which is which, the machine can be considered comparable to the human. The commonly human traits, such as the mind, consciousness or intelligence, should then extend to the machine in question. While the ability of rational speech is explicitly evaluated, the test also allows for evaluation of other properties of thinking that can be reported by language. Further, Turing's idea of learning machines as a possible solution to his test illustrates a connection between intelligence and learning.

Turing's paper [32] elicited many debates. Searle's [26] mind experiment with *the Chinese Room* can serve as a well known example. Actually, there are many problems with the original *Turing test*, be it in its stronger form as a test of intelligence or in its weaker form as a test of language. These problems originate in the purely language-based communication. Dennett [4] argues that since language reacts to the events happening in the world, as well as causes events to happen in the world, the *Turing test* is able to evaluate only a part of language capacity.

Apart from these language limits of *the Turing test*, Harnad [9] notes that the test sidelines a broad variety of human behavior that is commonly considered intelligent. By transforming Turing's question into the form: "Do machines have a mind?" Harnad links the evaluation of machine intelligence to the *Mind–Body*

Problem. In this broad set of philosophical problems, Harnad focuses on its epistemological form and picks a solution called the *Analogical Inference* (for an in-depth discussion consult [15]). Thus, according to Harnad, humans attribute minds to others based on their real word behavior that they find indistinquishable from their own behaviour in comparable situations. Consequently, Harnad proposes *the Total Turing test* in which the tested subject is a robot interacting with the real world. This setting would enable both the evaluation of language capabilities in relation to the world, as well as the evaluation of a broad set of other intelligent behavior.

However, even such an extension of the test is not quite sufficient, as can be shown by considering the philosophical theory of *Externalism*. According to this theory, see e.g. [18], there are several external aspects necessary for understanding language, due to its social, historical, and evolutionary origin. Employing this theory, Schweizer [25] proposes *the Truly Total Turing test* in which there is not a single tested subject but a particular species of subjects interacting with the world. The goal of the test is to ascertain that the tested species (robots) can evolve their own language and intelligent behavior, and not merely parasite on the pre-existing language and behavior of another species (such as that of humans).

The introduced approaches focus on the manifestation of thought–intelligence in language communication and real-world behavior and to some extent learning and understanding. For a sufficient evaluation of such capacities in machines it is, however, necessary to conduct a rather demanding *Truly Total Turing Test*. Even if this was possible, some fundamental objections would remain, for example, the dependence on a human interrogator, succeptibility of the test to cheating, or its Boolean result. For a more detailed account see [10].

2.2 Intelligence as a Cognitive Ability

As indicated by some approaches from Section 2.1, intelligence is not solely a property of the mind but related to other cognitive abilities. These are subject to cognitive science governed by a *cognitive paradigm* [3]. The main idea is that information processing requires the system in question to have a certain form of a *world model* for environment representation. There is likely not a single world model but a set of models of different granularity and structure that together form a dynamic cognitive schema to organize knowledge. The schema also forms expectations about the world that influence its *perception*, making it both object and subject dependent. The process of *knowledge acquisition* depends both on perception as well as on *action* and through these also on current knowledge.

An important part of cognitive science focuses on cognitive modelling. The resulting *cognitive architectures* are domain-generic computational models of structures and proceses in human or animal minds [29]. While their primary purpose is to test theories of cognitive science, they can also inspire AI development. Further, as these models aim to be general, they also require general evaluation.

Through *Psychometry*, psychology as a part of cognitive science has a long tradition of systematically measuring psychological properties (especially intelli-

gence) in humans and animals using various tests. According to Bringsjord and Shimanski [2], psychometry answers the question "What is intelligence?" and AI should thus be conceived as psychometric. This means that AI should strive to construct systems that would perform well in all established and validated tests of intelligence and other mental abilites. However, as Besold et al. [1] argues, it is problematic to adopt psychometric tests and definitions without any changes. It would be necessary to modify and, more importantly, generalize them.

Cognitive science can help in defining intelligence and putting it in context of other cognitive abilites. The level of details of such description can be rather high as shown by cognitive architectures. However, while directly employing this knowledge has its benefits it also has many limits as shown by psychometric AI.

3 Algorithmic Information Theory

The answers given in Section 2 to the question "What is intelligence?" are either rather abstract and informal or tightly connected to existing biological systems. More solid methods are needed to facilitate the development of AGI and measure its progress. In this section, a number of different approaches originating from *the Algorithmic Information Theory* will be discussed, including *the C-Test* (3.1), *the Universal Intelligence definition* (3.2), *the Pragmatic General Intelligence* and *the Efficient Pragmatic General Intelligence definitions* (3.3), *the Anytime Intelligence Test proposal* (3.4), *the Algorithmic Intelligence Quotient Test* (3.5), a formal *measure of environment difficulty* (3.6), and a formal *Task Theory* (3.7).

3.1 C-Test

In his pioneering work [10] from 2000, Hernández-Orallo proposes a formal computational measure of intelligence founded in *Algorithmic Information Theory*. The formalism focuses on a factor of intelligence that "allows us to comprehend the world." The resulting *C-Test* is a practically feasible test, yet a drawback of this approach is its sole focus on static tasks.

3.2 Universal Intelligence Definition

Intelligence is a high-level concept with complex connections to other similar concepts that are also difficult to grasp. According to Legg and Hutter [20], it is the lack of a formal definition of intelligence that hampers progress in the field of AI.

In order to keep the definition as broad as possible, Legg and Hutter [19] surveyed a variety of definitions, theories and tests of both human and animal intelligence. After generalizing the results, they arrived at the following informal definition: "Intelligence measures an agent's ability to achieve goals in a wide range of environments." A formalization of this working definition is given by Legg and Hutter [20] in Equation 1. The definition described by this equation is called *Universal Intelligence*.

Universal Intelligence [20] considers step-based interaction between the agent π (which sends actions a_i) and an environment μ (which sends rewards r_i and observations o_i). The definition does not restrict considered agents, however to facilitate the construction of a computer-run test, only environments that can be described by *Turing computable* probability measures are considered.

$$\Upsilon(\pi) := \sum_{\mu \in E} 2^{-K(\mu)} V_\mu^\pi, \text{ where } \qquad V_\mu^\pi := \mathbb{E}\left(\sum_{i=1}^{\infty} r_i\right) \leq 1, \tag{1}$$

where *Universal Intelligence* Υ of an agent π is given by its ability to achieve goals described by the value function V_μ^π as an expected sum of all future rewards over the set E of environments μ weighted by *Algorithmic Probability* that is based on *Kolmogorov complexity* K [17]. Consequently, an agent's success in various environments contributes to its intelligence to a varying degree: in accordance with *Occam's Razor*, complex environments have a lower impact than less complex ones.

Universal Intelligence [20] can be understood as a generalization of *C-Tests* [10] from static tasks to interactive dynamic environments. While this definition is very general and has several desirable properties, it also has several practical limitations. Its adoption of *Kolmogorov complexity* makes it uncomputable, which is further amplified by it considering all (i.e. infinitely many) Turing computable environments as well as infinitely long agent-environment interaction sequence. Thus, any derived test can only be an approximation.

While *Universal Intelligence* [20] is not culturally dependent or antropocentric, it does due to the *Kolmogorov complexity* dependend on the chosen *reference Turing machine*. This dependence can lead to serious issues, as shown by Hibbard [14], since it makes the overall measure biased towards, and dominated by, a relatively small set of environments described by short programs. The environments in the set can then become profoundly different when the reference machine is changed. To reduce this issue, a minimal length of programs can be set arbitrarily.

Hibbard [14] also proved that intelligence measures have to be based on unequal weighting of environments in order to avoid *the No Free Lunch Theorem*. This condition is met by *Universal Intelligence* since it employs *Algorithmic Probability* to weight the environments.

3.3 Pragmatic General Intelligence Definition

Universal Intelligence [20] has some limitations. Goertzel [7] notes further issues when considering real agents in real environments:

- In addition to the explicit goals given by the environment, there are also implicit goals originating in the agent, however no definition should make assumptions about the agent's cognitive architecture.
- Agents are usually adapted to certain environments (through evolution or design). Instead of true universality, a biased generality is of more interest when evaluating intelligence.

- Real agents in real environments are always bound by limited resources, therefore efficiency of intelligence is important.

Based on the above critique, Goertzel [7] introduces a definition of *Pragmatic General Intelligence*, as given by Equation 2:

$$\Pi(\pi) \equiv \sum_{\mu \in E, g \in G, T} \nu(\mu)\gamma(g,\mu)V^{\pi}_{\mu,g,T}, \tag{2}$$

where *the Pragmatic General Intelligence* Π of an agent π is given by its ability to achieve complex goals in complex environments as described by a value function $V^{\pi}_{\mu,g,T}$ as an expected sum of future rewards relative to a probability distribution ν of environments μ and a probability distribution γ of goals g in a time interval T.

Goertzel [7] modifies the value function: $V^{\pi}_{\mu,g,T} \equiv \mathbb{E}\left(\sum_{i=s}^{t} r_g(I_{g,s,i})\right)$, so that the agent's ability to achieve goals is related to a goal g given at the beginning of a time interval T in which the agent considers the goal related rewards r_g in all interaction sequences $I_{g,s,i}$ chosen according to the current environment μ.

In order to incorporate the computation efficiency, Goertzel [7] introduces the idea of *Efficient Pragmatic General Intelligence*, as defined in Equation 3:

$$\Pi_{\text{Eff}}(\pi) \equiv \sum_{\mu \in E, g \in G, Q, T} \frac{\nu(\mu)\gamma(g,\mu)\eta_{\pi,\mu,g,T}(Q)}{Q} V^{\pi}_{\mu,g,T}, \tag{3}$$

where $\eta_{\pi,\mu,g,T}$ describes the probability that an agent π in an environment μ while pursuing a goal g in a time interval T consumes Q units of computational resources. Since a probability distribution η is being used, agents can be non-deterministic. A positive real Q denotes an amalgam of time, memory, energy and other resources.

Since the probability distributions considered by Goertzel [7] are not restricted to Solomonoff–Levin *Universal Probability Distribution*, the sums in the equations need not converge and a search for convergence conditions is difficult.

Goertzel [7] also attempts to define an agent's generality, which he calls *Intelectual Breadth*. His approach uses a fuzzy set of contexts (triples of environment, goal and time interval) towards which the agent is intelligent. After normalizing the set into a probability distribution, the entropy is assessed. Yet Goertzel admits that his approach fails to consider interdependencies among environments and goals.

3.4 Anytime Intelligence Test

Hernández-Orallo and Dowe [13] proposed *the Anytime Intelligence Test* as a test of intelligence aimed at current and future artificial as well as biological agents of any intelligence level operating at any time scale. The test can be interrupted at any time, resulting in as precise an estimate of an agent's intelligence as the time allowed. The test combines a computatable modification of *Universal Intelligence* [20] with the prior work on *C-tests* [10] and *compression-enhanced Turing tests* [6].

Hernández-Orallo and Dowe [13] dealt with the three aspects of uncomputability of *Universal Intelligence* in the following way:

- A finite sample is used to approximate the infinite set of all environments. This raises the question of *discriminative power* so that the sampled environments meaningfully contribute to the intelligence evaluation. Hernández-Orallo and Dowe proposed considering only *reward-sensitive* environments, i.e. those where the choice of agent's action can always influence its rewards.
- A finite number of agent-environment interactions is considered in order to approximate the infinite interaction sequence. Therefore, a suitable way of combining rewards into the overall score is needed. Hernández-Orallo and Dowe suggested averaging the rewads according to the number of interactions. In addition, they also require the environments to be balanced, meaning that rewards are from an interval of $[-1, +1]$ and random behavior leads to an average close to 0. Such restrictions are necessary for this type of score-aggregation method to be meaningfull.
- A computable complexity function that is inspired by Levin's Kt Complexity [23] is used to approximate the *Kolmogorov Complexity*. The function is based on an upper bound on the computation time required for a single interaction and is further limited by the total number of interactions. Hernández-Orallo and Dowe call it $Kt^{\max}$. The function preserves *Occam's Razor* while also enforcing a time limit for the computation of an environment.

Hernández-Orallo and Dowe [13] also include physical time in the test. For an environment, this is achieved by the $Kt^{\max}$ function. For an agent, a time limit of the test run is proposed and the agent's reaction time can be included into the overall score. The number of interactions is no longer fixed, but depends on the time limit and agent's reaction time. The score averaging method also consideres delays between the agent's actions, discouraging it from cheating.

Hernández-Orallo and Dowe [13] propose the testing procedure to adapt the environment complexity and time limit to the agent's time scale. The initial environment complexity and time limit are low. The time limit increases if the agent fails to react in time. The complexity is increased when sufficiently high rewards are achieved, but decreased when sufficiently low rewards are achieved. Both mechanisms are balanced in order to discourage cheating.

Insa-Cabrera et al. [16] introduced a prototype implementation of *the Anytime Intelligence Test* and employed it in simple experiments. While this implementation used an interesting idea of species dependent interfaces in front of the same test, it remains an overly simplified version of the original proposal.

3.5 Algorithmic Intelligence Quotient Test

Legg and Veness [22] brought forward a practically feasible test that approximates *the Universal Intelligence definition* [20] and incorporates some of the ideas from *the Anytime Intelligence Test proposal* [13]. Legg and Veness transformed *the Universal Intelligence* described by Equation 1 into the form given in Equation 4. This resulting approximation is called *the Algorithmic Intelligence Quotient*.

To overcome the uncomputability of the original definition, *the Algorithmic Intelligence Quotient test (AIQ test)* [22] uses a finite *episode length* of k steps and a finite sample of N environment programs p_i that describe environments. Environment sampling maintains the idea of *Occam's Razor* while avoiding the *Kolmogorov Complexity* due to the choice of Solomonoff's *Universal Distribution* [27, 28]. Several programs in the sample can, however, describe the same environment.

$$\hat{\Upsilon}(\pi) := \frac{1}{N} \sum_{i=1}^{N} \hat{V}_{p_i}^{\pi}, \text{ where} \qquad \hat{V}_{p_i}^{\pi} := \frac{1}{k} \sum_{i=1}^{k} r_i, \tag{4}$$

where *the AIQ estimate of Universal Intelligence* $\hat{\Upsilon}$ of an agent π is given by its ability to achive goals as described by the empirical value function $\hat{V}_{p_i}^{\pi}$ as an average reward achieved by the agent over k interactions with an environment program p_i from a finite sample of N environment programs.

As is the case with *Universal Intelligence*, the choice of *reference machine* (the language of the environment programs) influences the classes of programs that are likely to be included in the sample [22]. To minimize this issue, *the AIQ test* uses a rather simple *BF reference machine* [24]. The BF is a very low-level language that only uses 10 instructions which are closely related to operating a Turing machine, yet the programs can be nondeterministic [21].

Honoring the call for *balanced environments* [13], the rewards are normalized to an interval $[-100, +100]$ which also limits the achivable AIQ score. The way the test works ensures that a randomly behaving agent will reach AIQ close to 0 [22, 21].

To deal with non-halting and long-running programs, a computation limit is set for an interaction. If the program tries to output more than a preconfigured number of observations, it is also halted. The proportion of non-interactive programs in the sample is reduced by excluding programs without read or write instructions and by excluding programs that return constant rewards [22, 21]. Thus, the call to exclude *environments without discriminative power* [13] is partially fulfiled.

Legg and Veness [22, 21] employed several variance reduction techniques to speed up the AIQ estimation. The implemented test is available as open source software. It can be configured in several ways: Setting the number of programs in the sample influences the precision of the estimate. Setting the episode length increases the "learning time" available to the agent. Changing the number of symbols used by the BF machine and the number of output observations influences the complexity of the interaction space. The implementation comes with a few supplied reinforcement learning agents, while others can be hooked in through a wrapper.

3.6 Environment Difficulty Measure

A novel view on environment difficulty was suggested by Hernández-Orallo [11, 12] that no longer identifies it with the complexity of environment description. The key idea of his approach is that it is actually the complexity of the solution that determins the difficulty of a problem or an environment.

Hernández-Orallo [11] explored several notions of difficulty functions. A version inspired by *Levin Complexity* [23] uses both the length of the solution as well as the number of computational steps required. According to this measure, the difficulty of an environment can then be given by the difficulty of its easiest solution.

The necessary changes in averaging an agent's performance over a range of tasks also reduce the bias resulting from the choice of reference machine [11]. First, there is a uniform or slightly decaying weighting of dificulties. Then, for a given difficulty, a set of solutions is derived that has a uniform distribution. Finally, for each solution, tasks are generated according to the *Universal Distribution*.

3.7 Task Theory

Tasks or environments play a central role in all introduced approaches to intelligence evaluation. Yet, the formalization of tasks focuses mainly on interfacing them to the evaluated agent and their overall properties such as complexity or difficulty. Thórisson at al. [30, 31] attempted to formulate a *Task Theory* to enable detailed understanding of tasks and ways to generate and compare them.

While work on *Task Theory* is still in the beginning, some initial results were given. A list of requirements for the theory was formulated [30, 31]. Since the final goal of AGI is to construct real-world agents, the main requirement is that the theory should be rooted in physics, and therefore enable modeling of physical tasks including the ensuing time and energy constraints.

An initial version of a formalism to facilitate task analysis was given [30, 31]. An environment can be described by a set of variables with associated domains, invariant relations between them, and dynamic functions to transform them over time from an initial state. An agent can perceive the environment through sensors with access to some of the variables, and act upon it through actuators with control over other variables. The task is given by a goal state with a set of desirable properties and restricted by a failure state with a set of undesirable properties.

4 Discussion and Conclusion

This paper overviewed approaches to evaluation of intelligence in artificial systems. A comparison of the introduced approaches identifies two distinct groups based on fundamentally different presumptions:

1. *Success in a complex task is a sufficient condition for intelligence.* A representative example is *the Turing test* [32]. While these approaches focus mainly on the level of complexity of the assigned task, the sufficiency of the condition can be challenged if entities different from humans are considered.

2. *Explicit verification of success in simple tasks, as well as in complex ones, is required for intelligence.* An example is *the AIQ test* [22]. This presumption makes such approaches suitable to evaluate intelligence not only in humans but also in entities such as artificial systems or animals.

While Section 2 also overviewed approaches to intelligence evaluation originating in philosophy and cognitive science, the main focus was on approaches employing *Algorithmic Information Theory* overviewed in Section 3. In contrast to the former, the later are usally thoroughly formalized, and are rooted in a strong theoretical background. Let us now briefly discuss the introduced approaches.

The C-Test [10] is an initial practical work in this area, however, its sole focus on static tasks is rather limiting. This issue was solved by *the Universal Intelligence definition* [20]. A shortcoming of the definition is its bias due to the choice of the *reference Turing machine* [20]. A further analysis of this issue including proposals to alleviate it was given in [14, 11]. A potential risk of Hibbard's solution [14] is that it reduces explicit testing of an agent in simple tasks which could undermine the key idea of *the Universal Intelligence definition*. Some caution is therefore needed when employing it. The critique of *Universal Intelligence* given by Goertzel [7] is not unfounded – intelligence evaluation cannot be fully decoupled from the real tasks that the evaluated systems are intended to solve. Nevertheless, evaluation methods originating in *Universal Intelligence* seem to be rather promissing.

Since *Universal Intelligence* is a definition, it is by itself insufficient. *The Anytime Intelligence Test proposal* [13] extends the original definition by interesting theoretical (relation of intelligence and time) as well as practical (adaptability) aspects, while *the Algorithmic Intelligence Quotient test* [22] remains closer to the original. There are, however, also practical considerations when choosing a suitable test. While *the Anytime Intelligence Test proposal* is supperior to *the AIQ test*, *the Anytime Intelligence Test prototype* is so simplified that *the AIQ test* is clearly the better method for general evaluation of intelligence in artificial systems. However as the paper showed, there are several limits:

- Since *the Universal Intelligence definition* depends on the choice of the *reference Turing machine* [20, 14, 11], it is also the case for *the AIQ test*. Further, this also holds for the parameters of the used BF machine [33]. However, a method proposed by Hibbard [14] was implemented in [33], that alleviated the issue.
- *The Universal Intelligence definition* considers the success of an agent in all computable environments. This set necessarily contains many *environments without discriminative power* as noted by [13]. While this is not really an issue for an intelligence definition, it is critical for a practical test, since it results in wasting resources as well as biasing the estimated score. As was confirmed practically in [34], *the AIQ test* is not guarded against some cases of non-discriminative environments. However, a method was implemented in [34] that alleviates the issue to a certain degree.
- *The Universal Intelligence measure* as well as the derived *AIQ score* include some of the aspects of intelligence such as the measure of success or generality, yet only do so implicitly, which makes detailed comparison of agents more complicated. Further, the effectiveness of the agent as advocated e.g. by [7] is not included at all, neither is the time aspect as advocated e.g. by [13].

Further practical limits of *the AIQ test* were analyzed in detail in [33, 34] resulting in several improvements of the test. Thus, *the AIQ test* remains the most suitable method for general evaluation of artificial systems that is both theoretically well grounded as well as readily available for practical use.

Many areas remain for future work in the field of general evaluation of artificial systems. These include:

- *A measure of environment dificulty*, proposed by Hernández-Orallo [11, 12], that could be compared to the way results are aggregated in *the AIQ test*.
- *A Task Theory* that is being pursued by Thórisson at al. [30, 31] could perhaps be used to gain a better understanding of the environment programs used in *the AIQ test* or even serve as a guide when generating new environments.

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Optimal design of production system under limited budget

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Abstract

In this paper we continue our previous research focused on multiple criteria optimization of system design. First we briefly recall Zeleny's basic model and two solution methods of so called de novo programming problems; namely, original Zeleny's approach and later modification proposed by Shi. Then we point out to some possible extensions and applications and present examples showing features that were not observed previously. We conclude with pointing to an interesting connection to the Nash model of cooperative bargaining.

1 Introduction

To avoid misunderstandings, we first agree on some notation and meaning of basic terms. Ordered n-tuples $x = (x_1, \ldots, x_n)$ of real numbers are considered as members of the real n-dimensional space $\mathbb{R}^n$. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ from $\mathbb{R}^n$ we write $x < y$ and $x \leq y$ if, respectively, $x_i < y_i$ and $x_i \leq y_i$ for each i from $\{1, 2, \ldots, n\}$. The relations $>$ and $\geq$ between n-tuples x and y are defined analogously. The sets $\{x \in \mathbb{R}^n : x \geq 0\}$ and $\{x \in \mathbb{R}^n : x > 0\}$ are denoted by $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. If S is a subset of $\mathbb{R}^n$ and x is a point in $\mathbb{R}^n$, then we denote the sets $\{a+x : a \in S\}$ and $\{a-x : a \in S\}$ by $S+x$ and $S-x$, respectively. Similarly, if λ is a real number, we define λS as the set $\{\lambda a : a \in S\}$. If a is such that $x \notin S$ for every $x > a$, then we say that a is *weakly Pareto optimal* in S. If a is such that $x \notin S$ for every $x \geq a$ with $x \neq a$, then we say that a is *strongly Pareto optimal* in S.

As a rule, we are not interested in solving only one particular example of an abstract problem but in being able to solve also many other possible examples. Therefore, when dealing with solution methods (procedures, algorithms), it is important to distinguish between "problems" and "instances of problems".

By analogy with the theory of computational complexity, we will understand by a *problem* a general question to be answered or a task to be solved that has several free parameters whose values are left unspecified. An *instance* of a problem is then given by specifying particular values for all the problem parameters. In other words, problems can be considered as sets of instances.

For example, when we consider the problem of finding a solution of system

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

of m linear equations over n variables, then an instance of this problem could be the task of finding a solution for the specific system of 2 linear equations in 3 variables

$$\begin{array}{rcl} 2x_1 + 5x_2 - 3x_3 & = & 5 \\ 3x_1 - 2x_2 + 2x_3 & = & 0 \end{array}$$

obtained by specifying values of parameters m, n, a_{ij}, b_i. Also it must be sufficiently clear what is meant by a solution. If the components of x are allowed to be arbitrary real numbers, then it would be a different problem if components of x are required to be integers.

We will be interested in studying a class of optimization problems. The basic ingredients of an optimization problem are (a) the set of instances or input objects, (b) the set of feasible solutions or output objects associated with any instance, and (c) the measure of quality of feasible solutions that is to be optimized. Formally, we define an optimization problem $\mathcal{O}$ as 4-tuple (I, F, M, G) where

- I is the set of instances of problem $\mathcal{O}$,
- F is a mapping that assigns to each instance $X \in I$ the set of feasible solutions of X,
- M is a function that assigns to every ordered pair (X, Y), where $X \in I$ and $Y \in F(X)$, some measure of quality of Y.
- G specifies the goal of optimization, which is usually[1] maximization or minimization. However, depending on the meaning of measure of quality it may need further specification.

[1] For example, when values of M are real numbers.

For example, in the case of maximization, the purpose of an optimization problem with respect to an instance X is to find a feasible solution $\hat{Y} \in F(X)$ such that

$$M(X, \hat{Y}) = \text{ maximum of } M(X, Y) \text{ subject to } Y \in F(X).$$

In this paper we are concerned with specially structured linear optimization problems that were introduced by Milan Zeleny under the name "De novo programming" or "De novo optimal system design"; see, for example, [4, 5, 6]. In this case, the optimization problem (I, F, M, G) we are interested in can be formulated as follows.

INSTANCE: A 4-tuple (A, B, C, p) where A is a real (m, n)-matrix, B is a positive number, C is a real (q, n)-matrix, and p is a real $(1, m)$-matrix p.

FEASIBLE SOLUTIONS: Ordered pairs (x, b) satisfying the system of inequalities

$$Ax - Eb \leq 0, pb \leq B, x \geq 0, y \geq 0$$

where E is the (m, m)-unit matrix.

MEASURE: $M(x, b) = Cx$.

GOAL: Maximization.

Here the goal needs further specification when $q > 1$ because in this case Cx is a vector in q-dimensional space, and it is not clear what is meant by maximum. Moreover, the set of feasible instances will be often restricted by further conditions on A, B, C, and p.

The paper is organized as follows. First we briefly recall Zeleny's basic model and two solution methods of de novo programming; namely, original Zeleny's approach [5] and Shi's modification [1]. Then we point out to some possible extensions and applications and present examples showing features that were not observed previously. We conclude with pointing to an interesting connection to the Nash model of cooperative bargaining and several remarks.

2 Zeleny's Model

To introduce Zeleny's approach to the optimal system design, we first recall one of the standard linear programming models for allocating resources to activities so as to attain a given economic objective; that is, the problem

$$\text{maximize } cx \text{ subject to } Ax \leq b \text{ and } x \geq 0,$$

where c is a given real $(1, n)$-matrix (vector), b a given real $(m, 1)$-matrix (vector), A is a given real (m, n)-matrix, and x is an $(n, 1)$-matrix (vector) of real variables.

This model have several natural interpretation. To be explicit we will have in mind the following meaning of input and output data: (i) For each activity j, x_j is

the decision variable whose values represent the levels of activity j, and c_j denotes the profit per unit of activity j. (ii) For each resource i, b_i is the available amount of resource i. (iii) For each resource i and each activity j, a_{ij} is the amount of resource i required to perform one unit of activity j.

In de novo programming, the components of b are considered to be real variables the values of which are restricted by the condition

$$pb \leq B,\ b \geq 0$$

where p is a given positive $(1, m)$-matrix (vector) whose components are interpreted as the unit prices of resources, and B is a given positive number representing the total available budget. It is worth mentioning that the linear programming problems in which there is some freedom in the choice of values of coefficients of an activity or levels of resources have been studied since the early days of linear programming; see, for example, the Chapter 22 (Programs with variable coefficients) in the famous book by Dantzig [13], or the Chapter 4 (Linear programming with set coefficients) in more recent book by M. Fiedler et al. [14].

To indicate that the components of b are now real variables we change the notation and use the letter y instead of b. Then the de novo problem of optimal system design can be formulated as the problem of

$$\begin{array}{ll} \text{maximizing} & cx + 0y \\ \text{subject to} & Ax - Ey \leq 0, py \leq B, x \geq 0, y \geq 0. \end{array}$$

We will see later that this specially structured linear programming problem can be solved very efficiently with the help of a continuous knapsack problem, provided all c_j and all $(pA)_j$ are positive.

The situation becomes more complex (and de novo programming more useful) when we have to deal with multiple criteria; that is, with the problem of

$$\begin{array}{ll} \text{maximizing} & Cx + 0y \\ \text{subject to} & Ax - Ey \leq 0, py \leq B, x \geq 0, y \geq 0, \end{array}$$

where C is a real (q, n)-matrix of coefficients of q objective functions. The complication is caused by the fact that finding an optimum requires the comparison of real vectors instead of comparison of real numbers.

2.1 Single criterion problems

We begin with de novo programming for single criterion problems because they are used in essential way for solving problems with multiple criteria. Thus let us consider the problem of

$$\text{maximizing} \qquad cx + 0y \tag{1}$$

$$\text{subject to} \qquad Ax - Ey \leq 0, py \leq B, x \geq 0, y \geq 0 \tag{2}$$

where we assume that every instance (A, B, C, p) is required to satisfy the conditions $B > 0,\ c > 0,\ p > 0,\ pA > 0$.

The instances of this problem can be solved by any of the standard procedures for linear programming but de novo programming solves them more efficiently by taking advantage of the special structure of the problem.

First we observe that each solution $(\bar{x}, \bar{y})$ of inequalities

$$Ax - Ey \leq 0, py \leq B, x \geq 0, y \geq 0 \tag{3}$$

is also a solution of inequalities

$$(pA)x \leq py, py \leq B, x \geq 0, y \geq 0.$$

It follows that if $(\bar{x}, \bar{y})$ is a solution of system (3), then $\bar{x}$ is a solution of system

$$(pA)x \leq B, x \geq 0. \tag{4}$$

Consequently, if $(\bar{x}, \bar{y})$ is an optimal solution of problem (1)-(2), then $\bar{x}$ is a feasible solution of the continuous knapsack problem

$$\begin{aligned} &\text{maximize} && cx && (5)\\ &\text{subject to} && (pA)x \leq B, x \geq 0, && (6) \end{aligned}$$

and $c\bar{x} \leq c\hat{x}$ where $c\hat{x}$ is an optimal solution of (5)-(6).

On the other hand, if $\hat{x}$ is an optimal solution of the knapsack problem (5)-(6), then (as we show later) there is a $\hat{y}$ such that $(\hat{x}, \hat{y})$ is a feasible solution of problem (1)-(2), and consequently we have $c\hat{x} = c\hat{x} + 0\hat{y} \leq c\bar{x} + 0\bar{y} = c\bar{x}$ where $(\bar{x}, \bar{y})$ is optimal solution for problem (1)-(2). Thus we can conclude that $c\hat{x} = c\bar{x} + 0\bar{y}$ whenever $\hat{x}$ and $(\bar{x}, \bar{y})$ is optimal for (5)-(6) and (1)-(2), respectively.

To solve the continuous knapsack problem, we first notice that the dual problem to

$$\begin{aligned} &\text{maximize} && c_1x_1 + c_2x_2 + \cdots + c_nx_n\\ &\text{subject to} && (pA)_1x_1 + \cdots + (pA)_nx_n \leq B\\ &&& x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0 \end{aligned}$$

has the form

$$\begin{aligned} &\text{minimize} && uB\\ &\text{subject to} && u \geq \tfrac{c_1}{(pA)_1}, \cdots, u \geq \tfrac{c_n}{(pA)_n}; u \geq 0. \end{aligned}$$

Let k be such that $\frac{c_k}{(pA)_k} = \max_j\{\frac{c_j}{(pA)_j}\}$ and define

$$\hat{u} = \frac{c_k}{(pA)_k},\ \hat{x}_k = \frac{B}{(pA)_k},\ \hat{x}_j = 0 \text{ if } j \neq k.$$

Then $\hat{x}$, $\hat{u}$ are not only feasible but also optimal because

$$c\hat{x} = \frac{c_kB}{(pA)_k}, \qquad \hat{u}B = \frac{c_kB}{(pA)_k}.$$

Now using the optimal solution $\hat{x}$ of the knapsack problem, we can obtain an optimal solution of problem (1)-(2) by using the following observations. First we observe that it follows from (2) that every feasible $(\hat{x}, y)$ must satisfy $y_i \geq a_{ik}\frac{B}{(pA)_k}$ for each $i \in \{1, 2, \ldots, m\}$. Consequently, if at least one of these inequalities would be satisfied strictly, then py would be greater than B and $(\hat{x}, y)$ would be infeasible. Therefore, we have to define $\hat{y}$ by setting $\hat{y}_i = a_{ik}\frac{B}{(pA)_k}$ for each i. Then $p\hat{y} = B$, and $(\hat{x}, \hat{y})$ is feasible, and therefore also optimal. Now it is also easy to see that $c\hat{x} = \frac{c_k B}{(pA)_k}$, $\hat{y} = A\hat{x}$, $B = (pA)\hat{x}$. The ordered triple $(\hat{x}, \hat{y}, \hat{z})$ where $\hat{z} = c\hat{x}$ is called the optimal system design [5].

2.2 Multiple criteria problems

De novo multiple criteria problems (I, F, M, G) differ from single criteria ones only in two items: (a) The matrices C in instances (A, B, C, p) have more than one row. (b) The values of measure M are real q-vectors with $q > 1$. As mentioned previously, the situation is more complex because it is not clear what is meant by goal G; that is, by maximization (or minimization). Since the relation $>$ between vectors is not a linear order, it is natural to understand by maximum (or minimum) a maximal (or minimal) element with respect the partial order $>$. Then the task of maximization is to find a feasible solution x whose value is a maximal (or minimal) element with respect to order $>$; that is, a Pareto optimal element. However, as a rule there are many maximal (or minimal) elements, and consequently many Pareto optimal solutions.

2.2.1 Zeleny's method

Zeleny [5] proposes to obtain a Pareto solution by the following procedure. Given an instance (A, B, C, p), we first solve all single criterion problems given by rows of matrix C; that is, the problems

$$\begin{array}{ll} \text{maximize} & c^\alpha x + 0y \\ \text{subject to} & Ax - Ey \leq 0, py \leq B, x \geq 0, y \geq 0 \end{array}$$

where $c^\alpha = (c_1^\alpha, c_2^\alpha, \ldots, c_n^\alpha)$ and $\alpha = 1, 2, \ldots, q$.

We know from the previous section that, for each α, $\bar{x}^\alpha$ solves the α-problem when the components of $\bar{x}^\alpha$ are defined by

$$\bar{x}_j^\alpha = \begin{cases} \frac{B}{(pA)_{k(\alpha)}} & \text{when } j = k(\alpha), \\ 0 & \text{when } j \neq k(\alpha), \end{cases}$$

where $k(\alpha)$ is such that $\frac{c^\alpha_{k(\alpha)}}{(pA)_{k(\alpha)}} = \max_j\{\frac{c_j^\alpha}{(pA)_j}\}$.

Let $z^* = (z_1^*, z_2^*, \ldots, z_q^*)$ where the components z_α^* are defined by $z_\alpha^* = c^\alpha \bar{x}^\alpha$, and let x^* be the optimal solution of problem

$$\text{minimize } (pA)x \text{ subject to } Cx = z^* \text{ and } x \geq 0. \tag{7}$$

Then the ordered triple

$$(\hat{x}, \hat{y}, \hat{z}) = \left(\frac{B}{B^*} x^*, \frac{B}{B^*} y^*, \frac{B}{B^*} z^* \right)$$

(where $B^* = (pA)x^*,\ y^* = Ax^*$) is called the *optimal design* with respect to B. In [5], the problem (7) is called the *meta-optimization* problem and value B^* is called the *meta-optimum* budget. It is useful to note that B^* is the minimum budget to achieve z^* by using x^* and y^*. Consequently, we cannot achieve z^* when $B^* > B$.

This method assumes that the system $Cx^* = z^*$ has a nonnegative solution, which is not guaranteed in general. To correct it, Shi [1] proposed to replace problem (7) by problem

$$\text{minimize } (pA)x \text{ subject to } Cx \geq z^* \text{ and } x \geq 0. \tag{8}$$

To distinguish it from Zeleny's meta-optimization problem (7) we refer to (8) as Shi's meta-optimization problem. It is of worth mentioning that even if (7) has a solution, we can sometimes obtain different optimal solution by solving (8) because optimal solutions of (7) are feasible but not necessarily optimal solutions of (8).

2.2.2 Shi's method

Analogously to Zeleny's aproach, Shi [1] first solves the individual problems

$$\begin{array}{ll} \text{maximize} & c^\alpha x + 0y \\ \text{subject to} & Ax - Ey \leq 0, py \leq B, x \geq 0, y \geq 0 \end{array}$$

by reducing them to the continuous knapsack problems. However, then the obtained solutions $\bar{x}^\alpha$ are used to construct another solution x^{**}, which he calls *synthetic optimal solution.* The description how to construct the synthetic solution allows two interpretations.

In one of them we construct x^{**} as simple combination of nonzero components of all different solutions $x^\alpha, \alpha = 1, 2, \ldots, \bar{q}$. If we arrange the objective functiosn appropriatly, then we can easily see that $\bar{q} \leq q$ and $\bar{q} \leq n$. We define the synthetic solution as the sum of these solutions; that is, $x^{**} = \sum_{\alpha=1}^{\bar{q}} x^\alpha$.

In another one we obtain x^{**} as the sum of all solutions $x^\alpha, \alpha = 1, 2, \ldots, q$; that is, we define the synthetic solution by $x^{**} = \sum_{\alpha=1}^{q} x^\alpha$. Here each individual solution is considered as many times as it maximizes the objective functions. So the synthetic solution actually includes different preference values for each objective functions. These weights of the individual objective functions are given by the multiplicity of occurrence of the respective optimal solutions x^α.

By using x^{**} we obtain as the optimal system design with respect B the ordered triple

$$(\hat{x}, \hat{y}, \hat{z}) = \left(\frac{B}{B^{**}} x^{**}, \frac{B}{B^{**}} y^{**}, \frac{B}{B^{**}} z^{**} \right)$$

where $y^{**} = Ax^{**},\ z^{**} = Cx^{**},\ B^{**} = (pA)x^{**}$.

3 De Novo and Cooperative Bargaining

To explain how the de novo programming problems are related to problems of cooperative bargaining, we begin with recalling the basic framework established by Nash [7] for studying bargaining situations.

In Nash's model the term bargaining refers to a situation in which n individuals (bargainers, players) either reach an agreement in a given set $\mathcal{A}$ of possible agreements, or fail to reach agreement. If no agreement is reached, then a given disagreement event D occurs. Each individual has a preference relation (a complete transitive relation) on the set $\mathcal{A} \cup \{D\}$, and it is required that each individual's preferences be defined not only on $\mathcal{A} \cup \{D\}$ but also on the set of lotteries, and that the assumptions of the von Neumann and Morgenstern utility theory are satisfied. Consequently, for each player i, there exists a utility function u_i. Given the set of possible agreements, the disagreement event, and utility functions for players' preferences, we can construct the set of all utility n-tuples that can be the outcomes of bargaining. As a result we obtain a subset S of $\mathbb{R}^n$ and a point d in S whose components represent players' utilities for the disagreement event D. It is important to notice that no restrictions are put directly on the set of possible agreements and that the same utility n-tuples may result from several combinations of agreements and utilities. Thus two different bargaining situations that induce the same pair (S, d) are not distinguished and are treated equally.

Using this framework, we can formulate the *n-player bargaining problem* as a nonempty collection $\mathcal{B}$ of pairs (S, d) where S is a nonempty subset of n-dimensional real linear space $\mathbb{R}^n$ and d is a point in S, see the Chapter 2 in [11] for details. Then a *solution function* on $\mathcal{B}$ (or simply a *solution* on $\mathcal{B}$) is a function f from $\mathcal{B}$ to $\mathbb{R}^n$ such that, for each instance (S, d) of $\mathcal{B}$, the value $f(S, d)$ of f belongs to S. A *solution outcome* for an instance (S, d) from $\mathcal{B}$ is the value $f(S, d)$ of f at (S, d).

These definitions are too general for having a meaningful theory. Therefore, the instances (S, d) forming a bargaining problem $\mathcal{B}$, and solution functions should have some reasonable properties. For example, in the Nash basic model, it is assumed that each instance (S, d) satisfies the following conditions: (i) Bargaining set S is compact and convex. (ii) Disagreement point d belongs to S. (iii) There is at least one point in S that is better than d for every player. There is an extensive game-theoretic literature on bargaining in which a multitude of solution functions are proposed and analyzed; see, for example, surveys [12] and [10].

The most interesting bargaining solutions for our purpose are those that depend essentially on the ideal point of bargaining set S. From among them, the Raiffa-Kalai-Smorodinsky (R-K-S) solution seems to be closely related to the de novo solutions proposed by Zeleny and Shi. The R-K-S solution [15] is defined as the function that assigns to every instance (S, d) the point of intersection of weakly Pareto points of S with the straight line joining the disagreement point and ideal point of S.

We can see this relationship clearly when we realize that: (i) For each nonnegative y satisfying $py \leq B$, the set $S(y) = \{z : z = Cx, Ax \leq y, x \geq 0\}$ together

with $d = 0$ form an instance of a bargaining problem. (ii) The vector z^* of the right-hand sides of equations or inequalities in the meta-optimization problem (7) or (8) are components of the ideal point of the set $S(y^*)$ where $y^* = Ax^*$. (iii) The point $\frac{B}{B^*}z^*$ is the point of intersection of the set of weekly Pareto optimal points in S with the straight line connecting z^* and $d = 0$.

Originally, the Nash model was studied mainly for the case of two players where the situation is often simpler. This is also true for relationship between de novo and bargaining. As an illustration we present several examples. These examples are concerned with various different bargaining situations where we define $\mathcal{A}(y)$ by

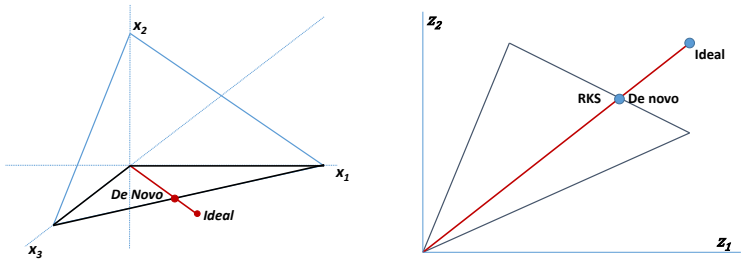
$$\mathcal{A}(y) = \{x \in \mathbb{R}^3 : x_1 + x_3 - y_1 \leq 0,\ x_2 - y_2 \leq 0,\ x \geq 0\}$$

for certain nonnegative y satisfying $2y_1 + 3y_2 \leq 12$; that is, $p = [2, 3]$ and $B = 12$. When we need to distinguish points resulting from Zeleny's and Shi's meta-problems, we use the notation $Z(x^*), S(x^*),\ Z(z^*), S(z^*)$ and so on.

Example 1. In this example we have two objective functions u_1, u_2 to be maximized which are defined by

$$u_1(x) = 2x_1 + x_2 + x_3,\ u_2(x) = 2x_1 + 4x_2 + 5x_3.$$

By solving the corresponding knapsack problems, we obtain as the ideal values $z_1^* = 12,\ z_2^* = 30$, and the solution of the corresponding meta-problem (7) is $x_1^* = 3.75,\ x_2^* = 0,\ x_3^* = 4.5$. It follows that $y_1^* = 8.25,\ y_2^* = 0$, and $B^* = 16.5$, which exceeds the available budget $B = 12$. Thus Zeleny's method delivers as optimal the system $x_1 + x_3 \leq 6,\ x_2 \leq 0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$, from which it selects the following Pareto optimal solution $\hat{x}_1 = 2.73,\ \hat{x}_2 = 0,\ \hat{x}_3 = 3.27$. Here the de novo solution is the same as R-K-S solution, see Figure 1.



(a) decisions (b) criteria

Figure 1: solutions

Notice that Shi's approach with x^{**} as the sum of both solutions will deliver solution $\hat{x}_1 = 3,\ \hat{x}_2 = 0,\ \hat{x}_3 = 3$.

Example 2. We wish to maximize the functions $u_1(x) = 3x_1 + x_2 + x_3,\ u_2(x) = x_1 + 4x_2 + 2x_3,\ u_3(x) = 2x_1 + x_2 + 4x_3$. Here, as previously the de novo and R-K-S

deliver the same results. The ideal point is $z^* = (18, 16, 24)$, the corresponding solution of both Zeleny's and Shi's meta-models is $x^* = (4.46, 1.14, 3.49)$, and $B^* = 19.31$, which exceeds the available budget $B = 12$. The intersection of the set of weakly Pareto points of S with the straight line joining the disagreement point 0 and ideal point is $\hat{x} = (2.77, 0.71, 2.17)$ with utilities $\hat{z} = (11.18, 9.94, 14.91)$. The corresponding components of $\hat{y}$ are $\hat{y}_1 = 4.94, \hat{y}_2 = 0.71$ and the prescribed budget 12 is respected.

EXAMPLE 3. Here we have the maximization of functions $u_1(x) = 2x_1 + x_2 + x_3$, $u_2(x) = x_1 + 4x_2 + 5x_3$, $u_3(x) = 3x_1 + x_2 + 2x_3$. The ideal point $Z(z^*) = (12, 30, 18)$ and the solution of Zeleny's meta-model is $Z(x^*) = (3, 3, 3)$, while Shi's meta-model has $S(z^*) = (12, 30, 20.67)$ as ideal point and $S(x^*) = (3.33, 0, 5.33)$ as solution of meta-problem. The intersection of S with the straight line joining the 0 and ideal point $Z(z^*)$ is $Z(\hat{x}) = (1.71, 1.71, 1.71)$ and $Z(\hat{z}) = (6.86, 17.14, 10.29)$, $Z(\hat{y}) = (3.43, 1.71)$, which is not a weakly Pareto point of S. With Shi's ideal point $S(z^*) = (12, 30, 20.67)$, we receive $S(x^*) = (2.31, 0, 3.69)$, and $S(\hat{z}) = (8.31, 20.77, 14.31), S(\hat{y}) = (6, 0)$. Shi's meta-model solution is an element of weakly Pareto points and dominates the previous solution. This result describes the behavior of R-K-S solutions between the situation where the R-K-S solution is a Pareto solution, and the situation where the R-K-S solution is equal to the disagreement point.

EXAMPLE 4. Here we have the maximization of functions $u_1(x) = 2x_1 + x_2 + x_3$, $u_2(x) = x_1 + 4x_2 + 5x_3$, $u_3(x) = 3x_1 + 6x_2 + 3x_3$. The ideal point is $Z(z^*) = (12, 30, 24)$. The Zeleny's meta-model has no solution. For Shi's meta-model we have $S(z^*) = (12, 30, 26)$ and $S(x^*) = (3.33, 0, 5.33)$. The intersection of set S with the straight line joining the disagreement point and ideal point is $x = (0, 0, 0)$ and $z = (0, 0, 0)$. With point $S(z^*)$, we receive $x = (2.31, 0, 3.69)$ and $z = (8.31, 20.77, 18), y = (6, 0)$.

We see from these examples that Zeleny's de novo solutions and R-K-S solutions coincides on two criteria instances but they can differ on instances with three criteria. Moreover, in the three criteria cases de novo provides reasonable results also in the cases where the R-K-S fails. This is caused by the well known fact that the R-K-S solutions requires that the instances (S, d) of bargaining problem $\mathcal{B}$ be comprehensive in the sense that y is necessarily in S whenever there is x in S such that $x \geq y$. We believe that the further analysis of this relationship is worth studying.

4 Remarks

REMARK 1. The main advantage of de novo programming in single criterion problems is that we obtain explicit dependence of solution on the input data. It makes it possible to redesign easily the optimal design when the budget is changed, or to solve easily some inverse optimization problems. For example, to determine the minimal budget for reaching a prescribed value of given

criterion.

REMARK 2. As pointed out by Zeleny in [5], in a competitive economy it would be more appropriate to maximize the difference $(cx-pb)$[2] than simply cx. Since the resources have prices in this model, they are not freely disposable. We can not only purchase them but also sell them. Thus we could sell the unused amount $b-Ax$. Then, it may be reasonable to maximize $(cx-pb)+p(b-Ax)$; that is, $(c-pA)x$, which does not require any change of method.

REMARK 3. The basic de novo model assumes that demand for production is unlimited. If there are bounds on values some or all variables x_i, then the reduction to the knapsack is again possible. Now the corresponding knapsack problem has the form

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & ax \le B,\ 0 \le x \le u \end{array}$$

where $a = pA$. If $au > B$, then we set $\hat{x} = u$. If $au \le B$, then we obtain $\hat{x}$ by the following procedure. Assume that $\frac{c_1}{a_1} \ge \frac{c_2}{a_2} \cdots \ge \frac{c_n}{a_n}$ and let k be such that

$$\sum_{j=1}^{k-1} a_j u_j < B \text{ and } \sum_{j=1}^{k} a_j u_j \ge B.$$

Then
$\hat{x}_i = u_i$ for $i \le k-1$, $\hat{x}_i = \frac{B-\sum_{j=1}^{k-1} a_j u_j}{a_k}$ for $i = k$, and $\hat{x}_i = 0$ otherwise.

REMARK 4. The De Novo methodology can be also adapted for models with capacity, requirement, and balance constraints [3]. The transformation to continuous knapsack problem is not possible in this case because the existence of a solution of the criteria constraints as equalities or inequalities is not guaranteed. Instead the knapsack problem the goal model is solved firstly and than the model minimizing the budget B for reachable criterion values is solved.

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Appendix

Table 1: approach (i)

Cl	Rule		dataset A		dataset B		
	R_id.	l	Supp.	Recall	Prec.	Recall	F 値
0-2	R_1^{0-2}	5	0.46	0.47	0.824	0.163	0.660
	R_2^{0-2}	3	0.22	0.23	0.882	0.366	0.840
	R_3^{0-2}	4	0.35	0.35	0.750	0.005	0.039
	R_4^{0-2}	5	0.31	0.31	0.909	0.414	0.842
	R_5^{0-2}	4	0.30	0.30	0.917	0.360	0.811
	R_6^{0-2}	1	0.05	0.06	0.950	0.1506	0.576
1-3	R_1^{1-3}	23	0.03	0.08	0.667	0.018	0.016
	R_2^{1-3}	11	0.03	0.09	0.475	0.056	0.0521
	R_3^{1-3}	16	0.03	0.08	0.611	0.032	0.0300
	R_4^{1-3}	21	0.03	0.08	0.438	0.021	0.019
	R_5^{1-3}	19	0.03	0.07	—	0.000	0.000
	R_6^{1-3}	18	0.03	0.06	0.300	0.009	0.008
	R_7^{1-3}	28	0.06	0.14	—	0.000	0.000
	R_8^{1-3}	14	0.03	0.09	1.000	0.006	0.005
	R_9^{1-3}	29	0.04	0.11	0.357	0.015	0.014
	R_{10}^{1-3}	24	0.04	0.11	0.529	0.027	0.025
	R_{11}^{1-3}	25	0.03	0.08	0.375	0.027	0.025
	R_{12}^{1-3}	15	0.03	0.08	0.539	0.021	0.019
	R_{13}^{1-3}	22	0.05	0.13	0.636	0.021	0.019
2-4	R_1^{2-4}	22	0.01	0.13	0.333	0.014	0.004
	R_2^{2-4}	26	0.02	0.15	0.000	0.000	0.000
	R_3^{2-4}	29	0.02	0.19	0.333	0.014	0.004
	R_4^{2-4}	29	0.02	0.16	0.500	0.007	0.003
	R_5^{2-4}	27	0.02	0.22	0.273	0.022	0.005
	R_6^{2-4}	27	0.02	0.22	0.200	0.007	0.002
	R_7^{2-4}	7	0.001	0.04	—	0.000	0.000
3-5	R_1^{3-5}	19	0.01	0.54	—	0.000	0.000
	R_2^{3-5}	19	0.01	0.46	0.500	0.015	0.002
4-6	R_1^{4-6}	7	0.001	1.00	—	0.000	0.000
14-18	R_1^{14-18}	2	0.001	1.00	—	0.000	0.000

Table 2: approach (ii)

Cl	Rule		dataset A		dataset B		
	R_id.	l	Supp.	Recall	Prec.	Recall	F 値
0-10	R_1^{0-10}	3	0.070	1.000	0.800	0.741	0.769
5-15	R_1^{5-15}	2	0.009	0.059	0.846	0.458	0.595
	R_2^{5-15}	2	0.143	0.929	0.370	0.917	0.527
	R_3^{5-15}	2	0.013	0.082	0.000	0.000	0.000
10-20	R_1^{10-20}	5	0.157	1.000	0.194	0.778	0.311
20-30	R_1^{20-30}	5	0.063	0.593	0.145	0.468	0.221
	R_2^{20-30}	2	0.043	0.407	—	0.000	0.000
30-40	R_1^{30-40}	6	0.099	0.917	0.191	0.354	0.248
	R_2^{30-40}	3	0.009	0.083	—	0.000	0.000
40-50	R_1^{40-50}	5	0.047	0.605	0.062	0.500	0.110
	R_2^{40-50}	3	0.031	0.395	—	0.000	0.000
50-60	R_1^{50-60}	4	0.047	0.619	0.000	0.000	0.000
	R_2^{50-60}	2	0.029	0.381	0.000	0.000	0.000
60-80	R_1^{60-80}	4	0.119	0.667	0.018	0.027	0.022
	R_2^{60-80}	2	0.060	0.333	0.516	0.219	0.308
80-100	R_1^{80-100}	4	0.063	0.833	0.243	0.920	0.385
	R_2^{80-100}	3	0.013	0.167	1.000	0.160	0.276
100-120	$R_1^{100-120}$	4	0.125	0.986	0.516	0.500	0.508
	$R_2^{100-120}$	2	0.027	0.214	—	0.000	0.000
120-	$R_1^{\geq 120}$	1	0.106	0.983	—	0.000	0.000
	$R_2^{\geq 120}$	3	0.002	0.017	—	0.000	0.000

Table 3: approach (iii)

Cl	Rule		dataset A		dataset B		
	R_id.	l	Supp.	Recall	Prec.	Recall	F 値
≤ 10	$R_1^{\leq 10}$	3	0.070	1.000	0.800	0.741	0.769
≤ 20	$R_1^{\leq 20}$	3	0.206	1.000	0.333	1.000	0.500
≤ 40	$R_1^{\leq 40}$	3	0.399	1.000	0.667	0.937	0.779
≤ 60	$R_1^{\leq 60}$	2	0.532	0.983	0.462	1.000	0.632
	$R_2^{\leq 60}$	4	0.294	0.543	0.355	0.187	0.245
≤ 80	$R_1^{\leq 80}$	3	0.708	0.997	0.635	0.912	0.749
	$R_2^{\leq 80}$	2	0.258	0.364	0.476	0.293	0.363
≤ 120	$R_1^{\leq 120}$	2	0.892	1.000	0.743	1.000	0.852
≥ 10	$R_1^{\geq 10}$	2	0.951	1.000	0.986	0.979	0.983
≥ 20	$R_1^{\geq 20}$	2	0.783	0.979	1.000	0.819	0.900
	$R_1^{\geq 20}$	2	0.0469	0.059	1.000	0.115	0.206
≥ 40	$R_1^{\geq 40}$	2	0.605	0.994	1.000	0.736	0.848
	$R_2^{\geq 40}$	3	0.0830	0.136	—	0.000	—
≥ 60	$R_1^{\geq 60}$	2	0.469	1.000	1.000	0.649	0.788
≥ 80	$R_1^{\geq 80}$	4	0.299	1.000	0.681	0.260	0.376
≥ 120	$R_1^{\geq 120}$	1	0.106	0.983	—	0.000	—
	$R_2^{\geq 120}$	3	0.002	0.017	—	0.000	—

Table 4: approach (iv)

Cl	Rule		dataset A		dataset B		
	R_id.	l	Supp.	Recall	Prec.	Recall	F 値
≤ 10	$R_1^{\leq 10}$	3	0.070	1.000	0.800	0.741	0.769
≤ 20	$R_1^{\leq 20}$	3	0.206	1.000	0.333	1.000	0.500
≤ 40	$R_1^{\leq 40}$	3	0.399	1.000	0.667	0.937	0.779
≤ 60	$R_1^{\leq 60}$	2	0.532	0.983	0.462	1.000	0.632
	$R_2^{\leq 60}$	1	0.487	0.900	0.634	1.000	0.776
≤ 80	$R_1^{\leq 80}$	3	0.708	0.997	0.635	0.912	0.749
	$R_2^{\leq 80}$	2	0.428	0.603	0.986	0.788	0.876
≤ 120	$R_1^{\leq 120}$	2	0.892	1.000	0.743	1.000	0.852
≥ 10	$R_1^{\geq 10}$	2	0.951	1.000	0.986	0.979	0.983
≥ 20	$R_1^{\geq 20}$	2	0.783	0.979	1.000	0.819	0.900
	$R_1^{\geq 20}$	2	0.545	0.682	0.971	0.662	0.787
≥ 40	$R_1^{\geq 40}$	2	0.605	0.994	1.000	0.737	0.848
	$R_1^{\geq 40}$	2	0.594	0.976	0.935	0.870	0.901
≥ 60	$R_1^{\geq 60}$	2	0.469	1.000	1.000	0.650	0.788
≥ 80	$R_1^{\geq 80}$	4	0.299	1.000	0.681	0.260	0.376
≥ 120	$R_1^{\geq 120}$	1	0.106	0.983	—	0.000	—
	$R_1^{\geq 120}$	2	0.002	0.017	0.935	0.210	0.343

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